automation

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Phase Preserving Balanced Truncation for Order Reduction of Positive Real Systems

Zeinab Salehi [1], Paknoosh Karimaghaee [1,*], Shabnam Salehi [2] and Mohammad-Hassan Khooban [3]

[1] Department of Electrical Engineering, Shiraz University, Shiraz 51154-71348, Iran; z.salehi@shirazu.ac.ir
[2] Department of Automation and Instrumentation, Petroleum University of Technology, Tehran 14539-63341, Iran; shnm.salehi@gmail.com
[3] Department of Electrical and Computer Engineering, Aarhus University, 8200 Aarhus, Denmark; khooban@eng.au.dk
* Correspondence: kaghaee@shirazu.ac.ir

Abstract: This paper presents a new passivity-preserving order reduction method for linear time-invariant passive systems, which are also called positive real (PR) systems, with the aid of the balanced truncation (BT) method. The proposed method stems from the conic positive real balanced truncation (CPRBT) method, which is a modification of the BT method for PR systems. CPRBT presents an algorithm in which the reduced models are obtained from some Riccati equations in which the phase angle of the transfer function has been taken into consideration. Although CPRBT is a powerful algorithm for obtaining accurate PR reduced-order models, it cannot guarantee that the phase diagram of the reduced model remains inside the same interval as that of the original full-order system. We aim to address such a problem by modifying CPRBT in the way that the phase angle of the reduced transfer function always remains inside the conic and homolographic phase interval of the original system. This is proven through some matrix manipulations, which has added mathematical value to the paper. Finally, in order to assess the efficacy of the proposed method, two numerical examples are simulated.

Keywords: model order reduction; balanced truncation; positive real systems; passivity; Riccati equations

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Citation: Salehi, Z.; Karimaghaee, P.; Salehi, S.; Khooban, M.-H. Phase Preserving Balanced Truncation for Order Reduction of Positive Real Systems. *Automation* **2022**, *3*, 84–94. https://doi.org/10.3390/automation3010004

Communicated by Hamid Reza Karimi

Received: 25 December 2021
Accepted: 17 January 2022
Published: 19 January 2022

1. Introduction

Model order reduction (MOR) plays a fundamental role in the analysis, control, and simulation of real-world systems. As most of these systems are naturally high order, a lot of computational effort is needed to analyze them. MOR tries to address this problem by simplifying the formulation of such high-order systems, creating reduced-order models with the desired accuracy. The input–output behavior of these reduced-order models is expected to be as close as possible to that of the original high-order systems. More importantly, MOR algorithms should guarantee that basic features of the original system, such as stability and passivity, remain unchanged in the reduced model.

Order reduction has been an interesting research topic since the 1980s. Therefore, a lot of order reduction methods were published during this period, some of which are balanced truncation (BT) [1], proper orthogonal decomposition (POD) [2], Hankel–Norm reduction [3], H_∞ model reduction [4], Padé Via Lanczos (PVL) [5], asymptotic waveform evaluation (AWE) [6,7], and PRIMA [8]. While these methods are designed to reduce the order of systems in all frequencies, some other methods are specifically developed to reduce the order of systems over a special frequency interval, such as Enns' [9] and Gawronski's [10] frequency-weighted methods. Several other model reduction procedures have been developed based on this concept including [11–13]. In addition, instead of reducing the order of systems in an infinite time horizon, some works consider reducing the order of systems over limited time intervals [14]. In order to find more publications in the area of model reduction, readers are referred to [15–18].

Among order reduction schemes, BT is so popular for reducing the order of linear time-invariant (LTI) systems, and it has a great reputation for preserving stability and offering an error bound. This method, however, cannot preserve some other features of the system such as passivity, bounded realness, and the negative imaginary property. In order to meet these requirements, some modifications to the BT method have been developed over the years such as bounded real balanced truncation (BRBT) [19], negative imaginary balanced truncation (NIBT) [20], positive real balanced truncation (PRBT) [21], mixed positive-bounded balanced truncation (MPBBT) [22], and conic positive real balanced truncation (CPRBT) [23].

Passive systems are a well-known class of dynamical systems that have wide applications in circuits, systems, and control theory [24–26]. The main feature of these systems is that they cannot produce energy, and they just store or dissipate energy [27]. Therefore, it is very important to maintain this property in the reduced models. In the case of LTI systems, passivity is also known as positive realness, and such systems are called positive real (PR) systems [27]. As mentioned before, the BT method cannot preserve passivity, and this is why some other modifications to this method have been developed which can preserve passivity, such as PRBT [21], mixed Gramian balanced truncation with error bounds [28], and CPRBT [23]. Some other passivity preserving order reduction methods are [22,29,30].

In this paper, we focus on the order reduction of PR systems whose transfer function lies inside a conic sector with an inner angle 2θ, as indicated in Figure 1, based on balancing methods. The phase angle of the transfer function gives us valuable information about the capacitive and inductive properties of that system. Using such information contributes to achieving more accurate reduced models. The CPRBT method is a passivity-preserving balancing-based method which has used the phase angle of the transfer function in order reduction for the first time [23]. This method, however, cannot guarantee that the transfer function of the reduced model remains in the conic sector with the inner angle 2θ displayed in Figure 1. In order to overcome this issue, this paper presents a novel order reduction scheme for PR systems that not only preserves passivity but also guarantees that the reduced model's phase angle lies inside $(-\theta, \theta)$ as shown in Figure 1. The proposed method, named "Phase Preserving Balanced Truncation", is inspired by the so-called CPRBT method and uses the same Riccati equations as CPRBT, but the choice of Gramians for balancing and order reduction is different. With the aid of these Gramians, a new realization is defined for PR systems, in which by selecting the states corresponding to the largest singular values, two parallel reduced models can be obtained. Separating these two parallel reduced models, we reach the final reduced model. Subsequently, it is proved via a theorem that not only the transfer function of the reduced model is passive, but it also remains in the conic sector with the inner angle 2θ. In order to assess the efficacy of the proposed method, two numerical examples are also provided, which confirm the mathematical results.

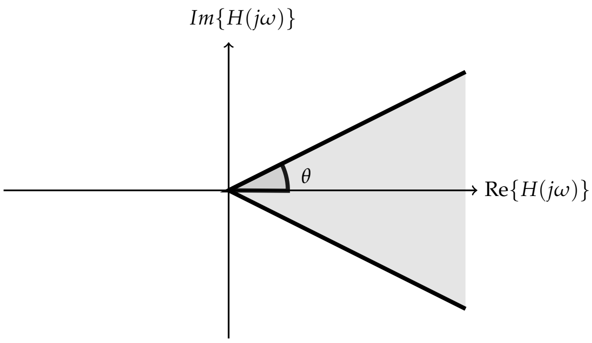

Figure 1. Conic sector with the inner angle 2θ.

The rest of this paper is arranged in the following sequence. The two order reduction methods, BT and CPRBT, are studied in Section 2. Then, a new order reduction method named "Phase Preserving Balanced Truncation" is introduced in Section 3. Subsequently, two numerical examples are included to assess the efficacy of the proposed method in Section 4. Finally, this paper ends with the conclusion in Section 5.

Notations: The fields of real and complex numbers are shown by symbols $\mathbb{R}$ and $\mathbb{C}$, respectively. The set of all $n \times m$ real matrices is represented by $\mathbb{R}^{n\times m}$. Furthermore, $\mathbf{I}_n$ denotes $n \times n$ identity matrices. The transpose of matrix $\mathbf{A}$ is shown by $\mathbf{A}^T$. Moreover, the inverse of $\mathbf{A}$ is represented by $\mathbf{A}^{-1}$, and the inverse of $\mathbf{A}^T$ is indicated by $\mathbf{A}^{-T}$. Moreover, $\mathbf{A} > 0$ $(\mathbf{A} \geq 0)$ means that matrix $\mathbf{A}$ is positive (semi) definite.

2. Preliminaries

This section describes two balancing-based order reduction methods: the BT method and the CPRBT method. In the first place, we assume that an asymptotically stable, minimal, and LTI system is represented by

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the state matrix, $\mathbf{B} \in \mathbb{R}^{n\times m}$ is the input matrix, $\mathbf{C} \in \mathbb{R}^{p\times n}$ is the output matrix, and $\mathbf{D} \in \mathbb{R}^{p\times m}$ is the feed-forward matrix. In addition, $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ is the input vector, and $\mathbf{y}(t) \in \mathbb{R}^p$ is the output vector of the system. The transfer function of this system is described by

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}. \tag{2}$$

2.1. Balanced Truncation (BT)

The main focus of the BT method is on finding a realization in which the controllability Gramian $\mathbf{P}$ and the observability Gramian $\mathbf{Q}$ are diagonal and equal in value [1]. For asymptotically stable and minimal systems, these two positive definite (PD) Gramians are the unique solutions of

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0, \tag{3}$$

$$\mathbf{A}^T\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{C}^T\mathbf{C} = 0. \tag{4}$$

The BT method finds a specific similarity transformation $\mathbf{T} \in \mathbb{R}^{n\times n}$ such that

$$\mathbf{T}^T\mathbf{Q}\mathbf{T} = \mathbf{T}^{-1}\mathbf{P}\mathbf{T}^{-T} = diag(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n > 0$. λ_i is known as the Hankel singular value of the system for $i = 1, \ldots, n$. Transforming the original system into the new coordination obtains

$$\mathbf{H}(s) \sim \left[\begin{array}{c|c} \hat{\mathbf{A}} & \hat{\mathbf{B}} \\ \hline \hat{\mathbf{C}} & \mathbf{D} \end{array}\right] = \left[\begin{array}{cc|c} \hat{\mathbf{A}}_{11} & \hat{\mathbf{A}}_{12} & \hat{\mathbf{B}}_1 \\ \hat{\mathbf{A}}_{21} & \hat{\mathbf{A}}_{22} & \hat{\mathbf{B}}_2 \\ \hline \hat{\mathbf{C}}_1 & \hat{\mathbf{C}}_2 & \mathbf{D} \end{array}\right]. \tag{5}$$

To reach the rth order reduced model $\mathbf{H}_r(s)$, BT selects the first r states which are associated with the r largest Hankel singular values; that is,

$$\mathbf{H}_r(s) \sim \begin{bmatrix} \hat{\mathbf{A}}_{11} & \hat{\mathbf{B}}_1 \\ \hat{\mathbf{C}}_1 & \mathbf{D} \end{bmatrix}.$$

The reduced model $\mathbf{H}_r(s)$, obtained by the BT method, is asymptotically stable and minimal [1,31]. Although this method preserves asymptotic stability, it is incapable of preserving passivity.

2.2. Conic Positive Real Balanced Truncation (CPRBT)

CPRBT is a balancing-based order reduction method for PR systems which preserve passivity. This method is the first one ever which has used the phase angle of the transfer function in order reduction to reach more accurate reduced models.

Definition 1 ([32]). *The transfer function* $\mathbf{H}(s)$ *in* (2) *is called positive real if*

$$m = p, \quad \mathbf{D}^T + \mathbf{D} \geq 0 \quad \textit{and} \quad \mathbf{H}^T(-j\omega) + \mathbf{H}(j\omega) \geq 0, \quad \textit{for} \quad \forall \omega \in \mathbb{R}. \tag{6}$$

Lemma 1 ([23]). *The transfer function of the positive real single-input single-output (SISO) system* $H(s)$ *lies in the conic sector* $(-\theta, \theta)$*, as shown in Figure 1, if and only if there exists* $\mathbf{F}_a = \mathbf{F}_a^T > 0 \in \mathbb{R}^{2n\times 2n}$ *such that*

$$\begin{aligned} \mathbf{A}_a^T\mathbf{F}_a + \mathbf{F}_a\mathbf{A}_a + \left(\mathbf{F}_a\mathbf{B}_a - \mathbf{C}_a^T\begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix}\right) \\ \times\mathbf{N}_a^{-1}\left(\mathbf{F}_a\mathbf{B}_a - \mathbf{C}_a^T\begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix}\right)^T = 0, \end{aligned} \tag{7}$$

where

$$\begin{aligned} \left[\begin{array}{c|c} \mathbf{A}_a & \mathbf{B}_a \\ \hline \mathbf{C}_a & \mathbf{D}_a \end{array}\right] := \left[\begin{array}{cc|cc} \mathbf{A} & 0 & \mathbf{B} & 0 \\ 0 & \mathbf{A} & 0 & \mathbf{B} \\ \hline \mathbf{C} & 0 & D & 0 \\ 0 & \mathbf{C} & 0 & D \end{array}\right], \\ \mathbf{N}_a := \mathbf{D}_a^T\begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix} + \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{bmatrix}\mathbf{D}_a, \end{aligned} \tag{8}$$

and $\mathbf{A}_a \in R^{2n\times 2n}$*,* $\mathbf{B}_a \in R^{2n\times 2}$*,* $\mathbf{C}_a \in \mathbb{R}^{2\times 2n}$*, and* $\mathbf{D}_a \in \mathbb{R}^{2\times 2}$*.*

The dual of (7) is given by

$$\begin{aligned} \mathbf{A}_a\mathbf{E}_a + \mathbf{E}_a\mathbf{A}_a^T + \left(\mathbf{E}_a\mathbf{C}_a^T - \mathbf{B}_a\begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix}\right) \\ \times\mathbf{N}_b^{-1}\left(\mathbf{E}_a\mathbf{C}_a^T - \mathbf{B}_a\begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix}\right)^T = 0, \end{aligned} \tag{9}$$

where $\mathbf{E}_a = \mathbf{E}_a^T > 0 \in \mathbb{R}^{2n\times 2n}$ and

$$\mathbf{N}_b := \mathbf{D}_a\begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix} + \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{bmatrix}\mathbf{D}_a^T. \tag{10}$$

We assume that $\mathbf{N}_a$ and $\mathbf{N}_b$ in (8) and (10) are nonsingular matrices. Equations (7) and (9) have two extremal solutions, which means any solutions $\mathbf{F}_a$ and $\mathbf{E}_a$ of (7) and (9), respectively, satisfy $0 < \mathbf{F}_{a_{min}} \leq \mathbf{F}_a \leq \mathbf{F}_{a_{max}}$ and $0 < \mathbf{E}_{a_{min}} \leq \mathbf{E}_a \leq \mathbf{E}_{a_{max}}$ [23]. For the rest of this paper by $\mathbf{F}_a$ and $\mathbf{E}_a$, we mean $\mathbf{F}_{a_{min}}$ and $\mathbf{E}_{a_{min}}$, respectively.

CPRBT partitions $\mathbf{F}_a$ and $\mathbf{E}_a$ into four blocks as

$$\mathbf{F}_a = \begin{bmatrix} \mathbf{F} & \mathbf{W} \\ \mathbf{W}^T & \bar{\mathbf{F}} \end{bmatrix}, \qquad \mathbf{E}_a = \begin{bmatrix} \mathbf{E} & \mathbf{V} \\ \mathbf{V}^T & \bar{\mathbf{E}} \end{bmatrix}, \tag{11}$$

where $\mathbf{F}, \bar{\mathbf{F}}, \mathbf{W} \in R^{n\times n}$ and $\mathbf{E}, \bar{\mathbf{E}}, \mathbf{V} \in \mathbb{R}^{n\times n}$. Following this, CPRBT balances the two Gramians $\mathbf{F}$ and $\mathbf{E}$, which are in the upper-left corner of $\mathbf{F}_a$ and $\mathbf{E}_a$, respectively, by finding a similarity transformation $\mathbf{T} \in \mathbb{R}^{n\times n}$ such that

$$\mathbf{T}^T\mathbf{F}\mathbf{T} = \mathbf{T}^{-1}\mathbf{E}\mathbf{T}^{-T} = \boldsymbol{\Delta} = diag(\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2),$$

where $\mathbf{\Delta}_1 = diag(\delta_1, \ldots, \delta_r)$, $\mathbf{\Delta}_2 = diag(\delta_{r+1}, \ldots, \delta_n)$, and $\delta_1 \geqslant \delta_2 \geqslant \ldots \geqslant \delta_n > 0$. Applying the similarity transformation $\mathbf{T}$ to the original system, one obtains

$$H(s) \sim \left[\begin{array}{c|c} \hat{\mathbf{A}} & \hat{\mathbf{B}} \\ \hline \hat{\mathbf{C}} & D \end{array}\right] = \left[\begin{array}{cc|c} \hat{\mathbf{A}}_{11} & \hat{\mathbf{A}}_{12} & \hat{\mathbf{B}}_1 \\ \hat{\mathbf{A}}_{21} & \hat{\mathbf{A}}_{22} & \hat{\mathbf{B}}_2 \\ \hline \hat{\mathbf{C}}_1 & \hat{\mathbf{C}}_2 & D \end{array}\right]. \tag{12}$$

Finally, CPRBT offers the reduced model $H_r(s)$ by choosing

$$H_r(s) \sim \left[\begin{array}{c|c} \hat{\mathbf{A}}_{11} & \hat{\mathbf{B}}_1 \\ \hline \hat{\mathbf{C}}_1 & D \end{array}\right]. \tag{13}$$

It is proved that the reduced model $H_r(s)$ obtained from CPRBT is positive real (passive) [23]. However, this method cannot guarantee that the phase angle of the reduced model remains inside $(-\theta, \theta)$.

Remark 1 ([23])**.** *CPRBT obtains the same results for positive real multiple-input multiple-output (MIMO) systems. For more information please refer to Remark 4 in [23].*

3. Phase Preserving Balanced Truncation

In this section, a new order reduction algorithm is presented for PR systems which not only preserves passivity but also guarantees that the phase angle of the reduced model remains inside $(-\theta, \theta)$, as shown in Figure 1. This means that the upper bound of the absolute value of the reduced model's phase angle would not be bigger than that of the original system's phase angle.

For the purpose of simplification, we consider SISO systems; however, the results are applicable to MIMO systems as well. Consider Riccati Equations (7) and (9). Unlike CPRBT which looks for a similarity transformation $\mathbf{T} \in \mathbb{R}^{n\times n}$ which balances $\mathbf{F} \in \mathbb{R}^{n\times n}$ and $\mathbf{E} \in \mathbb{R}^{n\times n}$ of (11), the proposed method searches for a similarity transformation $\mathbf{T}_a \in \mathbb{R}^{2n\times 2n}$ which balances $\mathbf{F}_a \in \mathbb{R}^{2n\times 2n}$ and $\mathbf{E}_a \in \mathbb{R}^{2n\times 2n}$ of (7) and (9) such that

$$\mathbf{T}_a^T \mathbf{F}_a \mathbf{T}_a = \mathbf{T}_a^{-1} \mathbf{E}_a \mathbf{T}_a^{-T} = \mathbf{\Psi} = diag(\mathbf{\Psi}_1, \mathbf{\Psi}_2),$$

where $\mathbf{\Psi}_1 = diag(\psi_1, \ldots, \psi_{2r})$, $\mathbf{\Psi}_2 = diag(\psi_{2r+1}, \ldots, \psi_{2n})$, and $\psi_1 \geqslant \psi_2 \geqslant \ldots \geqslant \psi_{2n} > 0$. We call ψ_i as the phase preserving singular value of the system for $i = 1, \ldots, 2n$.

Remark 2. *Note that the above statement is valid because when we apply the similarity transformation $\mathbf{T}_a$ to the system $(\mathbf{A}_a, \mathbf{B}_a, \mathbf{C}_a, \mathbf{D}_a)$, the Gramians $\mathbf{F}_a$ and $\mathbf{E}_a$ change based on the rules $\mathbf{F}_a \to \mathbf{T}_a^T \mathbf{F}_a \mathbf{T}_a$ and $\mathbf{E}_a \to \mathbf{T}_a^{-1} \mathbf{E}_a \mathbf{T}_a^{-T}$ [23]. Therefore, $\mathbf{E}_a \mathbf{F}_a \to \mathbf{T}_a^{-1} \mathbf{E}_a \mathbf{F}_a \mathbf{T}_a$, which means that the eigenvalues of $\mathbf{E}_a \mathbf{F}_a$ remain invariant under similarity transformation [23]. As a consequence, there exists a similarity transformation $\mathbf{T}_a$ which balances $\mathbf{F}_a$ and $\mathbf{E}_a$. It is clear that such a similarity transformation can be obtained in the same way as the BT method.*

Applying the similarity transformation $\mathbf{T}_a \in \mathbb{R}^{2n\times 2n}$ to the parallel system $H(s)\mathbf{I}_2$, one obtains

$$H(s)\mathbf{I}_2 = \begin{bmatrix} H(s) & 0 \\ 0 & H(s) \end{bmatrix} \sim \begin{bmatrix} \mathbf{T}_a^{-1}\mathbf{A}_a\mathbf{T}_a & \mathbf{T}_a^{-1}\mathbf{B}_a \\ \mathbf{C}_a\mathbf{T}_a & \mathbf{D}_a \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{A}}_a & \hat{\mathbf{B}}_a \\ \hat{\mathbf{C}}_a & \mathbf{D}_a \end{bmatrix}. \tag{14}$$

Then, partitioning $\hat{\mathbf{A}}_a, \hat{\mathbf{B}}_a, \hat{\mathbf{C}}_a$ based on the $2r$ largest ψ_i obtains

$$H(s)\mathbf{I}_2 \sim \left[\begin{array}{c|c} \hat{\mathbf{A}}_a & \hat{\mathbf{B}}_a \\ \hline \hat{\mathbf{C}}_a & \mathbf{D}_a \end{array}\right] = \left[\begin{array}{cc|c} \hat{\mathbf{A}}_{a_{11}} & \hat{\mathbf{A}}_{a_{12}} & \hat{\mathbf{B}}_{a_1} \\ \hat{\mathbf{A}}_{a_{21}} & \hat{\mathbf{A}}_{a_{22}} & \hat{\mathbf{B}}_{a_2} \\ \hline \hat{\mathbf{C}}_{a_1} & \hat{\mathbf{C}}_{a_2} & \mathbf{D}_a \end{array}\right]. \tag{15}$$

Now, we select the parallel reduced model $H_r(s)\mathbf{I}_2$ of order $2r$ by selecting the states corresponding to the $2r$ largest ψ_i as

$$\begin{bmatrix} H_r(s) & 0 \\ 0 & H_r(s) \end{bmatrix} \sim \left[\begin{array}{c|c} \hat{\mathbf{A}}_{a_{11}} & \hat{\mathbf{B}}_{a_1} \\ \hline \hat{\mathbf{C}}_{a_1} & \mathbf{D}_a \end{array}\right]. \tag{16}$$

Since $H_r(s)\mathbf{I}_2$ represents two isolated parallel systems, it is so straightforward to extract the reduced model $H_r(s)$. For the sake of clarification, the algorithmic steps of the proposed method are summarized in Algorithm 1.

Algorithm 1 Phase preserving balanced truncation

1: Consider the positive real and SISO system $H(s)$ whose phase angle is inside $(-\theta, \theta)$.
2: Compute $\mathbf{F}_a \in \mathbb{R}^{2n\times 2n}$ and $\mathbf{E}_a \in \mathbb{R}^{2n\times 2n}$ by solving (7) and (9), respectively.
3: Obtain the balancing similarity transformation $\mathbf{T}_a \in \mathbb{R}^{2n\times 2n}$ such that

$$\mathbf{T}_a^T\mathbf{F}_a\mathbf{T}_a = \mathbf{T}_a^{-1}\mathbf{E}_a\mathbf{T}_a^{-T} = \mathbf{\Psi} = diag(\mathbf{\Psi}_1, \mathbf{\Psi}_2),$$

where $\mathbf{\Psi}_1 = diag(\psi_1, \ldots, \psi_{2r})$, $\mathbf{\Psi}_2 = diag(\psi_{2r+1}, \ldots, \psi_{2n})$, and $\psi_1 \geqslant \psi_2 \geqslant \ldots \geqslant \psi_{2n} > 0$.
4: Find the state space form of $H(s)\mathbf{I}_2$ in the new coordination as

$$H(s)\mathbf{I}_2 = \begin{bmatrix} H(s) & 0 \\ 0 & H(s) \end{bmatrix} \sim \begin{bmatrix} \mathbf{T}_a^{-1}\mathbf{A}_a\mathbf{T}_a & \mathbf{T}_a^{-1}\mathbf{B}_a \\ \mathbf{C}_a\mathbf{T}_a & \mathbf{D}_a \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{A}}_a & \hat{\mathbf{B}}_a \\ \hat{\mathbf{C}}_a & \mathbf{D}_a \end{bmatrix}. \tag{17}$$

5: Partition $\hat{\mathbf{A}}_a$, $\hat{\mathbf{B}}_a$, $\hat{\mathbf{C}}_a$ based on the $2r$ largest ψ_i as

$$H(s)\mathbf{I}_2 \sim \left[\begin{array}{c|c} \hat{\mathbf{A}}_a & \hat{\mathbf{B}}_a \\ \hline \hat{\mathbf{C}}_a & \mathbf{D}_a \end{array}\right] = \left[\begin{array}{cc|c} \hat{\mathbf{A}}_{a_{11}} & \hat{\mathbf{A}}_{a_{12}} & \hat{\mathbf{B}}_{a_1} \\ \hat{\mathbf{A}}_{a_{21}} & \hat{\mathbf{A}}_{a_{22}} & \hat{\mathbf{B}}_{a_2} \\ \hline \hat{\mathbf{C}}_{a_1} & \hat{\mathbf{C}}_{a_2} & \mathbf{D}_a \end{array}\right]. \tag{18}$$

6: Select $H_r(s)\mathbf{I}_2$ as

$$\begin{bmatrix} H_r(s) & 0 \\ 0 & H_r(s) \end{bmatrix} \sim \left[\begin{array}{c|c} \hat{\mathbf{A}}_{a_{11}} & \hat{\mathbf{B}}_{a_1} \\ \hline \hat{\mathbf{C}}_{a_1} & \mathbf{D}_a \end{array}\right]. \tag{19}$$

7: As $H_r(s)\mathbf{I}_2$ represents two separate systems, extract the rth order $H_r(s)$ from (19) by eliminating the redundant states.

Theorem 1. *The phase diagram of the reduced model $H_r(s)$ obtained from the proposed algorithm always remains inside $(-\theta, \theta)$.*

Proof. Assume that the system $H(s)\mathbf{I}_2$ is in the balanced realization (17). The Riccati equation (7) in the new coordination would be

$$\begin{aligned} &\hat{\mathbf{A}}_a^T\mathbf{\Psi} + \mathbf{\Psi}\hat{\mathbf{A}}_a + \left(\mathbf{\Psi}\hat{\mathbf{B}}_a - \hat{\mathbf{C}}_a^T\begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix}\right) \\ &\times\mathbf{N}_a^{-1}\left(\mathbf{\Psi}\hat{\mathbf{B}}_a - \hat{\mathbf{C}}_a^T\begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix}\right)^T = 0. \end{aligned} \tag{20}$$

Considering $\mathbf{\Psi} = diag(\mathbf{\Psi}_1, \mathbf{\Psi}_2)$, we can extract the $(1,1)$ block of (20) as

$$\begin{aligned} &\hat{\mathbf{A}}_{a_{11}}^T\mathbf{\Psi}_1 + \mathbf{\Psi}_1\hat{\mathbf{A}}_{a_{11}} + \left(\mathbf{\Psi}_1\hat{\mathbf{B}}_{a_1} - \hat{\mathbf{C}}_{a_1}^T\begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix}\right) \\ &\times\mathbf{N}_a^{-1}\left(\mathbf{\Psi}_1\hat{\mathbf{B}}_{a_1} - \hat{\mathbf{C}}_{a_1}^T\begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix}\right)^T = 0. \end{aligned} \tag{21}$$

Since $(\hat{\mathbf{A}}_{a_{11}}, \hat{\mathbf{B}}_{a_1}, \hat{\mathbf{C}}_{a_1}, \mathbf{D}_a)$ represents two separate parallel systems, there exists a similarity transformation $\bar{\mathbf{T}}_a \in \mathbb{R}^{2r\times 2r}$ which can extract these two systems as

$$\left[\begin{array}{c|c} \bar{\mathbf{A}}_a & \bar{\mathbf{B}}_a \\ \hline \bar{\mathbf{C}}_a & \mathbf{D}_a \end{array}\right] := \left[\begin{array}{c|c} \bar{\mathbf{T}}_a^{-1}\hat{\mathbf{A}}_{a_{11}}\bar{\mathbf{T}}_a & \bar{\mathbf{T}}_a^{-1}\hat{\mathbf{B}}_{a_1} \\ \hline \hat{\mathbf{C}}_{a_1}\bar{\mathbf{T}}_a & \mathbf{D}_a \end{array}\right] = \left[\begin{array}{cc|cc} \mathbf{A}_r & 0 & \mathbf{B}_r & 0 \\ 0 & \mathbf{A}_r & 0 & \mathbf{B}_r \\ \hline \mathbf{C}_r & 0 & D & 0 \\ 0 & \mathbf{C}_r & 0 & D \end{array}\right]. \tag{22}$$

Applying $\bar{\mathbf{T}}_a$ to the system $(\hat{\mathbf{A}}_{a_{11}}, \hat{\mathbf{B}}_{a_1}, \hat{\mathbf{C}}_{a_1}, \mathbf{D}_a)$ also yields

$$\begin{aligned} &\bar{\mathbf{A}}_a^T\bar{\mathbf{\Psi}} + \bar{\mathbf{\Psi}}\bar{\mathbf{A}}_a + \left(\bar{\mathbf{\Psi}}\bar{\mathbf{B}}_a - \bar{\mathbf{C}}_a^T\begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix}\right) \\ &\times\mathbf{N}_a^{-1}\left(\bar{\mathbf{\Psi}}\bar{\mathbf{B}}_a - \bar{\mathbf{C}}_a^T\begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix}\right)^T = 0, \end{aligned} \tag{23}$$

where $\bar{\mathbf{\Psi}} = \bar{\mathbf{T}}_a^T\mathbf{\Psi}_1\bar{\mathbf{T}}_a$. In the meantime, because of the fact that $\mathbf{\Psi}_1 > 0$, we yield $\bar{\mathbf{\Psi}} > 0$. As a consequence, based on the Riccati equation (23), we can say Lemma 1 holds for the reduced-order model $H_r(s)$ with the state-space form

$$H_r(s) \sim \left[\begin{array}{c|c} \mathbf{A}_r & \mathbf{B}_r \\ \hline \mathbf{C}_r & D \end{array}\right]. \tag{24}$$

This completes the proof. □

Corollary 1. *The reduced model $H_r(s)$ obtained from the proposed algorithm preserves passivity.*

Remark 3. *The same results can be held for positive real MIMO systems although we have only studied SISO systems for simplification. To this end, we have to use the associated Riccati equations for MIMO systems which are defined in [23] instead of (7) and (9) and then use the same procedure as the proposed algorithm to obtain the reduced model.*

4. Illustrative Examples

This section provides two illustrative examples to investigate the efficacy of the proposed method.

4.1. Example 1

In this example, a 10th order PR system is examined whose state-space form is

$$\begin{aligned} &\mathbf{A} = \underset{k=1,\dots,5}{block\,diag}\begin{pmatrix} 0 & 1 \\ -\pi_k^2 & -2\mu\pi_k \end{pmatrix}, \quad \pi_k = k^2, \quad k = 1,\dots,5, \quad \mu = 0.01, \\ &\mathbf{C} = \mathbf{B}^T = [\; 0 \quad 0.9877 \quad 0 \quad -0.309 \quad 0 \quad -0.891 \quad 0 \quad 0.5878 \quad 0 \quad 0.7071 \;], \\ &D = 0.2. \end{aligned} \tag{25}$$

We want to reach the reduced model of order $r = 4$ utilizing two methods: CPRBT and the proposed method (phase preserving balanced truncation). As the phase angle of the transfer function is inside $(-83, 83)$, we chose $\theta = 83$ in both methods. The state-space form of the 4th order reduced model obtained by the proposed method is given by

$$\left[\begin{array}{c|c} \mathbf{A}_r & \mathbf{B}_r \\ \hline \mathbf{C}_r & D \end{array}\right] = \left[\begin{array}{cccc|c} -0.0423 & 2.6976 & 0.0036 & -0.0151 & 1.0438 \\ -2.6027 & -0.0545 & -1.3232 & 0.0217 & -1.2909 \\ -0.0399 & 1.3668 & -0.0104 & -1.1544 & 0.2793 \\ 0.0389 & -0.0508 & 1.1624 & -0.0122 & -1.1533 \\ \hline 0.5282 & -0.6426 & 0.1335 & -0.5749 & 0.2 \end{array}\right]. \tag{26}$$

The bode diagrams of the reduced-order models, along with the full-order system, are depicted in Figure 2.

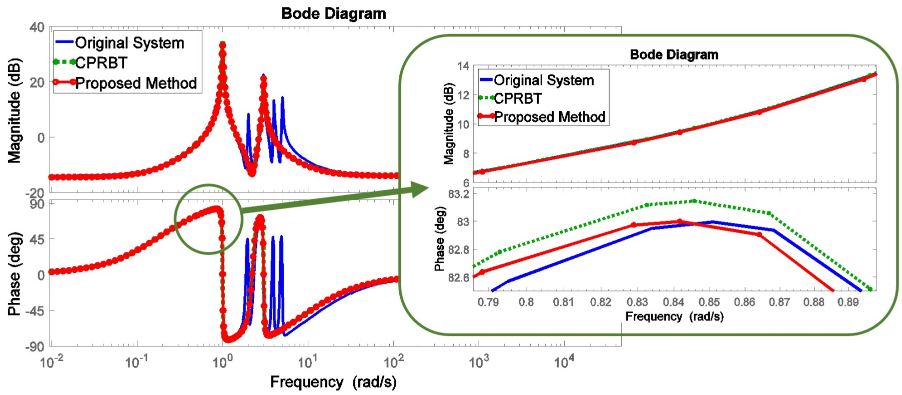

Figure 2. The bode diagrams of the original system, the CPRBT reduced model, and the proposed reduced model in Example 1.

Comparing the phase angle of the reduced models, we observe that the phase angle of the proposed reduced model remains inside $(-83, 83)$, while the phase angle of the reduced model obtained by CPRBT exceeds this bound and comes inside $(-83.145, 83.145)$. This clearly shows the efficacy of the proposed method. The two reduced models approximately have the same levels of accuracy in terms of the input–output behavior of the system, and their magnitude in the bode plot almost matches each other. Figure 2 completely confirms the results.

4.2. Example 2

Here, we provide an example which illustrates the effectiveness of the proposed method regarding accuracy in the input–output behavior of the system. Consider the 6th order PR system with the state-space form

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 100 & -100 & 0 \\ 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & -110 & 0 & 0 & -100 \\ -100 & 0 & 0 & -10 & 0 & 100 \\ 100 & -100 & 0 & 0 & -10 & 0 \\ 0 & 0 & 100 & -100 & 0 & 0 \end{bmatrix}, \tag{27}$$
$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 100 & 0 & 0 & 0 \end{bmatrix}^T, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad D = 1.$$

The phase angle of the above system is inside $(-18.1, 18.1)$. Therefore, we choose $\theta = 18.1$ for the proposed method and CPRBT. We also select $r = 4$ to reach the 4th order reduced models. The reduced model obtained by the proposed method is given by

$$\left[\begin{array}{c|c} \mathbf{A}_r & \mathbf{B}_r \\ \hline \mathbf{C}_r & D \end{array}\right] = \left[\begin{array}{cccc|c} -100.34 & -108.79 & 14.35 & 23.06 & -13.22 \\ 111.56 & -3.42 & -59.26 & -12.07 & 0.65 \\ -62.33 & 64.26 & -7.20 & -32.65 & -3.37 \\ 50.91 & 13.71 & 48.68 & -14.16 & 4.67 \\ \hline -6.40 & -0.18 & -1.52 & 2.50 & 1 \end{array}\right]. \tag{28}$$

The bode diagrams of the reduced models, along with the full-order system, are illustrated in Figure 3. In addition, the frequency response of the error system $H(s) - H_r(s)$, obtained from CPRBT and the proposed method, is depicted in Figure 4 which enables us to compare the results precisely.

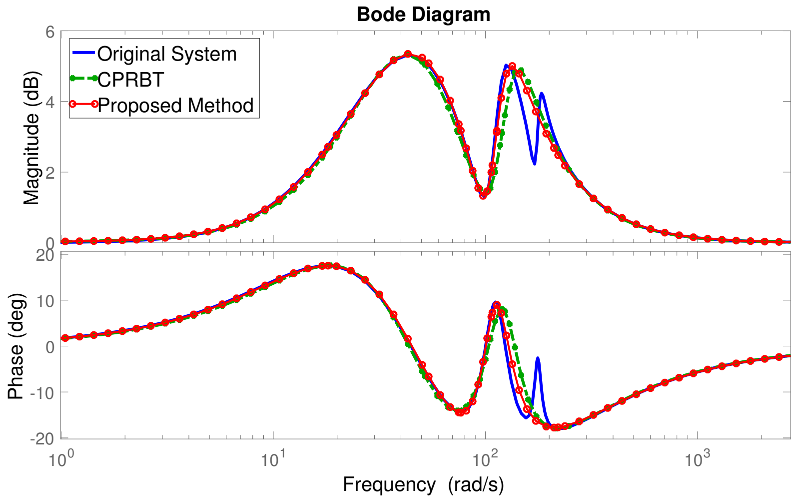

Figure 3. The bode diagrams of the original system, the CPRBT reduced model, and the proposed reduced model in Example 2.

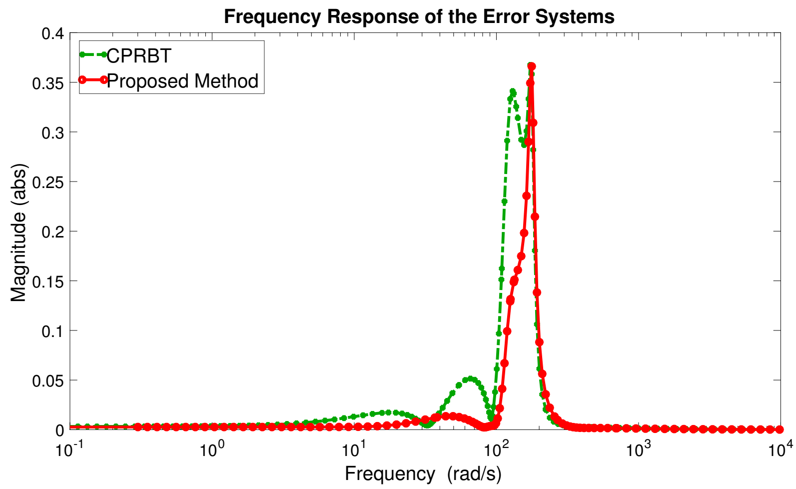

Figure 4. The frequency response of the error system $H(s) - H_r(s)$ obtained from CPRBT and the proposed method in Example 2.

As shown in Figures 3 and 4, the proposed method creates a more accurate reduced model than CPRBT, although both methods provide reduced models whose phase angles remain inside $(-18.1, 18.1)$. The H_∞ norm of the error system $H(s) - H_r(s)$ is 0.3714 for the CPRBT reduced model and 0.3685 for the proposed reduced model, which confirms that the proposed method has offered a better reduced-model than CPRBT.

5. Conclusions

This paper proposes a new passivity-preserving order reduction method for PR systems whose phase angle is inside the interval $(-\theta, \theta)$. The proposed method is a modification to the CPRBT method. What makes this method different from CPRBT is that it proves the phase angle of the reduced models remains inside the same interval as that of the original system, in contrast to CPRBT. In addition, in some cases, the proposed method can produce more accurate reduced models than CPRBT. In order to assess this method, two numerical examples are simulated. Example 1 perfectly verifies that the phase angle of the reduced model obtained by the proposed method remains inside $(-\theta, \theta)$, while the reduced model obtained by CPRBT exceeds this interval. In addition, Example 2 shows that the proposed method can even find reduced models of higher accuracy than CPRBT. These two points show the effectiveness of the proposed method. Although the provided examples are simple, the proposed method can be applied to high-dimensional systems as well. As future work, it is of interest to find a computational H_∞ error bound for the proposed method. It is also suggested to extend this method to discrete-time, nonlinear, and descriptor systems. Furthermore, the robustness of the proposed method can be the topic of future research.

Author Contributions: Conceptualization, Z.S., P.K., S.S., and M.-H.K.; methodology, Z.S., P.K., S.S., and M.-H.K.; software, Z.S.; validation, Z.S. and P.K.; formal analysis, P.K. and M.-H.K.; investigation, Z.S. and S.S.; resources, Z.S., P.K., S.S., and M.-H.K.; writing—original draft preparation, Z.S. and S.S.; writing—review and editing, P.K. and M.-H.K.; visualization, Z.S. and P.K.; supervision, P.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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