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The Orbital Graph of Primitive Group with Socle $A_7 \times A_7$

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Abstract

In this paper, we mainly study the orbital graphs of primitive groups with the socle $A_7 \times A_7$ which acts by diagonal action. Firstly, we calculate the element conjugate classes of A_7, then we discuss the stabilizer of two points in A_7. Finally, according to the relation between suborbit and orbital, we obtain the orbitals, and determine the orbital graphs.

Keywords

Primitive Group, Simple Diagonal Action, Suborbit, Orbital Graph

1. Introduction

Let G be a primitive group with the socle $A_7 \times A_7$, and $T = A_7$, $H = \{(t,t) \mid t \in T\}$. G acts on $\Omega = \{Hx \mid x \in A_7\}$. The stabilizer of α in G is G_α, α^G is the orbit of α under *G*. For any $\alpha \in G$, we call the orbit of G_α is suborbit of *G*, and the length of the suborbit is called subdegree. Let G be a group acting transitively on a set *Ω*. Then G induces a natural action on the Cartesian product $\Omega \times \Omega$. The orbits of G on this set are called the orbitals of G on *Ω*. For each orbital Δ, we can denote an orbital Δ', where $(\alpha,\beta) \in \Delta'$ if and only if $(\beta,\alpha) \in \Delta$. We call Δ and Δ' are paired orbitals. Clearly, $(\Delta')' = \Delta$. An orbital Δ is called self-paired if $\Delta' = \Delta$.

There is a close relationship between the orbitals of G and the orbits of G_α. For each orbital Δ of G and each $\alpha \in \Omega$, we define $\Delta(\alpha) := \{\beta \in \Omega \mid (\alpha,\beta) \in \Delta\}$ which is the set of points which are in the same orbit with α under G_α. It is easy to verify that the mapping $\Delta \mapsto \Delta(\alpha)$ is a bijection from the set of orbitals of G onto the set of orbits of G_α with the diagonal orbital mapping onto the trivial orbit $\{\alpha\}$. In particular, the number of orbits is equal to the number of orbits of G_α, this number is called the rank of *G*.

Usually, we depart graphs into two types: directed graph (digraph) and undirected graph. A digraph Γ is a paired (V,E), where E is a subset of $V\times V$. V represents the vertex set and E represents the edge set. If $(\alpha,\beta)\in E$ whenever $(\beta,\alpha)\in E$, then the digraph is called undirected. For a given orbital, there is an induced orbital graph. We see the elements of Ω as vertices and the orbital Δ of G as edge, then we will obtain a diagraph Graph (Δ). Obviously, the action of G on Graph (Δ) induces an arc-transitive action. Further, if Δ is self-paired, then the corresponding graph is an undirected arc-transitive graph.

There are many papers which refer to the suborbits of primitive group. In 1964, Wielandt's book referred that if G is a finite primitive group with a suborbit of length 2, then G is a dihedral group of order $2q$ (q prime) [1]. In 1967, Wong successively determined all primitive groups with a suborbit of length 3 and length 4 [2]. About the orbital, Quirin studied the primitive permutation groups with small orbitals in 1971 [3]. Later, Liebeck and Saxl determined the finite primitive permutation groups of rank three in 1986 [4], and on point stabilizers in primitive permutation groups in 1991 [5]. In 2004, Li, Lu and Marušič gave a complete classification of primitive permutation groups with small suborbits [6], they also determined the orbital graphs of such groups, in particular, for the valency 3 and 4. In the book [7], Xu proved that the diagonal subgroup of $T\times T$, where T is a nonabelian simple group, is the unique maximal subgroup in $T\times T$. He also calculated the lengths of the suborbits of $A_5\times A_5$. In 2014, the author has analyzed the orbitals of primitive group with socle $A_6\times A_6$ by SD action and PA action [8]. In 2017, doctor Wu Cixuan discussed the orbital graphs of finite permutation groups and their relevant edge-transitive graphs [9].

Here we just discuss the primitive group with SD type. According to the classification of finite simple group and the O'Nan-Scott theorem, we acquire the following theorem.

Theorem 1.1 Let $G=A_7\times A_7$, $H=\{(t,t)\mid t\in A_7\}$, $A_7\cong H\leq G$. Consider the right multiplicative action of G on $\Omega=[G:H]$. Then the information of the orbitals of G on Ω is listed in the following table:

i	K_i	l_i	Self-Paired (Y/N)
1	A_7	1	Y
2	S_4	105	N
3	$Z_3^2\times Z_4$	36	Y
4	$Z_3\times Z_3$	9	Y
5	$Z_2\times Z_2$	4	Y
6	A_4	12	Y
7	Z_5	5	Y
8	Z_7	7	N
9	Z_7	7	N

where in the table, K_i represents the stabilizer of the point stabilizer H on an orbit; l_i represents the length of suborbit;

Because the order of A_7 is 2520, which is a little large. Also the number of subgroups of A_7 is more than A_5 and A_6 and the structure of subgroup is very complex. So discuss the orbital graph of $A_7 \times A_7$ is a challenge job. In this paper, we depart the elements of A_7 by conjugation in **Lemma 3.1** firstly, and determine the length of suborbits, then we obtain the stabilizers of two points in A_7 in **Lemma 3.2**. Last, we determine the orbital by **Lemma 2.4** and obtain the final result. Besides, the method has advantage for the discussion of some complex group, and based on the result of this paper, we can discuss the structure of orbital graph further.

2. Basic Concepts of Permutation Groups

In this paper, about the primitive groups and orbit, we need know the theorems as following.

Lemma 2.1 Let G be a group, $a \in G$, then the number of the conjugated elements of a in G is $\left|G : N_G(a)\right|$.

Lemma 2.2 [7] Let T be a non-abelian simple group, $G = T \times T$. Let $H = \{(t,t) \mid t \in T\}$. Then H is maximal subgroup of G, and so the action of G on the set $\Omega = [G : H]$ is primitive.

Lemma 2.3 [7] Let $G = T \times T$, and T be a nonabelian simple group. Let $H = \{(t,t) \mid t \in T\}$ and G act primitively on the set Ω of right cosets of H in G. Then the length of suborbit equals to the length of conjugated class in T.

Lemma 2.4 [6] Let G be a group acting transitively on a set Ω, Δ be an orbital of G on Ω, and $(\alpha,\beta) \in \Delta$. Then Δ is self-paired if and only if there exists $g \in G$ such that $(\alpha,\beta)^g = (\beta,\alpha)$. In this case, we have $g^2 \in G_\alpha$, $g \in N_G(G_{\alpha\beta}) \setminus G_\alpha$. In particular, $\left|N_G(G_{\alpha\beta}) : G_{\alpha\beta}\right|$ is even.

3. The Orbital Graphs of Primitive Group with Socle $A_7 \times A_7$

According to the Latex, we know the maximal subgroups of A_7 are A_6, $L_2(7)^A$, $L_2(7)^B$, S_5, $(A_4 \times Z_3) : Z_2$, so we can know all subgroups of A_7. Here we just discuss the suborbits of G with SD type. Firstly, we give the calculation of the suborbits of G with SD type.

Lemma 3.1 Let $T = A_7$. Dividing the elements in A_7 by conjugation, then we have the following result:

1) The element (1) forms a conjugate class, denote it by C_1, then $C_1 = (1)$;

2) The elements of type $(12)(34)$ form a conjugate class, denote it by C_2. Then $C_2 = (12)(34)^{A_7}$, and the length is 105;

3) The elements of type (123) form a conjugate class, denoted it by C_3. Then $C_3 = (123)^{A_7}$, and the length is 70;

4) The elements of type $(123)(456)$ form a conjugate class, denoted it by C_4. Then $C_4 = (123)(456)^{A_7}$, and the length is 280;

5) The elements of type $(12)(3456)$ form a conjugate class, denoted it by C_5.

Then $C_5=(12)(3456)^{A_7}$, and the length is 630;

6) The elements of type $(123)(45)(67)$ form a conjugate class, denoted it by C_6. Then $C_6=(123)(45)(67)^{A_7}$, and the length is 210;

7) The elements of type (12345) form a conjugate class, denoted it by C_7. Then $C_7=(12345)^{A_7}$, and the length is 504;

8) The elements of type (1234567) form two conjugate classes, denoted them by C_8 and C_9. Then $C_8=(1234567)^{A_7}$ and the length is 360;
$C_9=(1234576)^{A_7}$ and the length also is 360.

Proof. $C_1=(1)$, obviously, $|C_1|=1$. For $C_2=(12)(34)^{A_7}$, since $(12)(34)=(21)(43)$ and for arbitrary two elements α,β, there exist $c\in A_7$ such that $\alpha^c=\beta$, so $|C_2|=\dfrac{\frac{A_7^2}{2}\times\frac{A_5^2}{2}}{2}=105$.

Similar, for $C_3=(123)^{A_7}$, $|C_3|=\dfrac{A_7^3}{3}=70$.

For $C_4=(123)(456)^{A_7}$, $|C_4|=\dfrac{\frac{A_7^3}{3}\times\frac{A_4^3}{3}}{2}=280$.

For $C_5=(12)(3456)^{A_7}$, $|C_5|=\dfrac{A_7^2}{2}\times\dfrac{A_5^4}{4}=630$.

For $C_6=(123)(45)(67)^{A_7}$, $|C_6|=\dfrac{A_7^3}{3}\times\dfrac{\frac{A_4^2}{2}\times\frac{A_2^2}{2}}{2}=210$.

For $C_7=(12345)^{A_7}$, $|C_7|=\dfrac{A_7^5}{5}=504$.

Because $(1234567)\times(67)=(1234576)$, but (67) does not belong in A_7, so they form two different conjugate classes.

For $C_8=(1234567)^{A_7}$, $|C_8|=\dfrac{A_7^7}{7}=360$.

For $C_9=(1234576)^{A_7}$, $|C_9|=\dfrac{A_7^7}{7}=360$.

According to **Lemma 2.3**, the length of suborbits of $A_7\times A_7$ acts on $[G:H]$ are 1, 105, 70, 280, 630, 210, 504, 360 and 360. We denote them by $l_1,l_2,l_3,l_4,l_5,l_6,l_7,l_8,l_9$.

Now we calculate the order of two-point stabilizer. Let $G=A_7\times A_7$ acts primitively on $\Omega=\{Hx\mid x\in A_7\}$, where $H=\{(t,t)\mid t\in T=A_7\}$. $G_\alpha\cong H$ is the stabilizer of α in *G*. For arbitrary $\beta\in\Omega$, by **Lemma 2.3**,

$\left|\beta^{G_\alpha}\right|=\left|G_\alpha:G_{\alpha\beta}\right|=\left|H:G_{\alpha\beta}\right|=\left|T:N_T(t)\right|$. Then $\left|G_{\alpha\beta}\right|=\dfrac{|G_\alpha|}{\left|\beta^{G_\alpha}\right|}=\dfrac{|T|}{|C_i|}=\left|N_T(t_i)\right|$.

So let $T=A_6$ acts on $\Delta=\{t\mid t\in A_6\}$ by conjugation, then:

1) If $t_1=(1)$, then $\left|G_{\alpha\beta_1}\right|=\left|N_T(t_1)\right|=\dfrac{2520}{1}=2520$;

2) If $t_2=(12)(34)$, then $\left|G_{\alpha\beta_2}\right|=\left|N_T(t_2)\right|=\dfrac{2520}{105}=24$;

3) If $t_3 = (123)$, then $\left|G_{\alpha\beta_3}\right| = \left|N_T(t_3)\right| = \frac{2520}{70} = 36$;

4) If $t_4 = (123)(456)$, then $\left|G_{\alpha\beta_4}\right| = \left|N_T(t_4)\right| = \frac{2520}{280} = 9$;

5) If $t_5 = (12)(3456)$, then $\left|G_{\alpha\beta_5}\right| = \left|N_T(t_5)\right| = \frac{2520}{630} = 4$;

6) If $t_6 = (123)(45)(67)$, then $\left|G_{\alpha\beta_6}\right| = \left|N_T(t_6)\right| = \frac{2520}{210} = 12$;

7) If $t_7 = (12345)$, then $\left|G_{\alpha\beta_7}\right| = \left|N_T(t_7)\right| = \frac{2520}{504} = 5$;

8) If $t_8 = (1234567)$, then $\left|G_{\alpha\beta_8}\right| = \left|N_T(t_8)\right| = \frac{2520}{360} = 7$;

9) If $t_9 = (1234576)$, then $\left|G_{\alpha\beta_9}\right| = \left|N_T(t_9)\right| = \frac{2520}{360} = 7$.

According to the order of $G_{\alpha\beta}$, we can determine the two-point stabilizer subgroups. Then we have the following result:

Lemma 3.2 Let $G = A_7 \times A_7$, $A_7 \cong H = G_\alpha$, $C_i\,(i = 2,3,4,5,6,7,8,9)$ be the conjugate class in A_7. Then:

1) For $C_1 = (1)$, there is $G_{\alpha\beta_1} \cong A_7$;
2) For $C_2 = (12)(34)$, there is $G_{\alpha\beta_2} \cong S_4$;
3) For $C_3 = (123)$, there is $G_{\alpha\beta_3} \cong Z_3^2 \times Z_4$;
4) For $C_4 = (123)(456)$, there is $G_{\alpha\beta_4} \cong Z_3 \times Z_3$;
5) For $C_5 = (12)(3456)$, there is $G_{\alpha\beta_5} \cong Z_2 \times Z_2$;
6) For $C_6 = (123)(45)(67)$, there is $G_{\alpha\beta_6} \cong A_4$;
7) For $C_7 = (12345)$, there is $G_{\alpha\beta_7} \cong Z_5$;
8) For $C_8 = (1234567)$, there is $G_{\alpha\beta_8} \cong Z_7$;
9) For $C_9 = (1234576)$, there is $G_{\alpha\beta_9} \cong Z_7$.

In order to know the suborbits are self-paired or not, by **Lemma 2.4**, we need to calculate the $N_G(G_{\alpha\beta})$. Let $G = T \times T$, $H = \{(t,t) \mid t \in T\}$, $K^* = \{(k,k) \mid k \in K \text{ and } K \le T\}$, then $K \cong K^* \le H \cong T$. Take $(k,k) \in K^*$, $g = (x,y) \in N_G(K^*)(x, y \in N_T(K) = N_H(K))$, we have: $(k,k)^{(x,y)} = (k^x, k^y) \in K^*$. So $k^x = k^y$, $k^{xy^{-1}} = k$, $xy^{-1} \in N_H(k) = C_H(k) \subseteq C_H(K)$. There exists $c \in C_H(K)$ such that $xy^{-1} = c$, $x = cy$, then *g* can be described as (cy, y), so $N_G(K^*) = \{(cy,y) \mid c \in C_H(K)$ and $y \in N_H(K)$ and $\left|N_G(k^*)\right| = \left|C_H(K)\right| \cdot \left|N_H(K)\right|$.

Let $K = G_{\alpha\beta}$, we have:

1) $G_{\alpha\beta} = A_7$.

Obviously, $\left|N_G(A_7)\right| = \left|C_H(A_7)\right| \cdot \left|N_H(A_7)\right| = |1| \cdot |A_7| = 2520$.

2) $G_{\alpha\beta} = S_4$.

By GAP, we know $N_H(S_4) = (A_4 \times Z_3):Z_2$, $C_H(S_4) = Z_3$, so $\left|N_G(S_4)\right| = \left|C_H(S_4)\right| \cdot \left|N_H(S_4)\right| = \left|Z_3\right| \cdot \left|(A_4 \times Z_3):Z_2\right| = 72$.

3) $G_{\alpha\beta} = Z_3^2 \times Z_4$.

Because $Z_3^2 \times Z_4$ is the maximal subgroup of A_6 and A_6 is the subgroup of A_7, $N_H\left(Z_3^2 \times Z_4\right) = Z_3^2 \times Z_4$. $C_H\left(Z_3^2 \times Z_4\right) = Z_3^2 \times Z_4$. So:

$\left|N_G\left(Z_3^2 \times Z_4\right)\right| = \left|C_H\left(Z_3^2 \times Z_4\right)\right| \cdot \left|N_H\left(Z_3^2 \times Z_4\right)\right| = \left|Z_3^2 \times Z_4\right| \cdot \left|Z_3^2 \times Z_4\right| = 1296$.

4) $G_{\alpha\beta} = Z_3 \times Z_3$.

Because $Z_3 \times Z_3 \lhd \left(Z_3 \times Z_3\right) : Z_4$ and $\left(Z_3 \times Z_3\right) : Z_4$ is maximal in A_6, $N_H\left(Z_3 \times Z_3\right) = \left(Z_3 \times Z_3\right) : Z_4$. And also because Z_2 and Z_4 don't centralize $Z_3 \times Z_3$, $C_H\left(Z_3 \times Z_3\right) = Z_3 \times Z_3$. So:

$\left|N_G\left(Z_3 \times Z_3\right)\right| = \left|C_H\left(Z_3 \times Z_3\right)\right| \cdot \left|N_H\left(Z_3 \times Z_3\right)\right| = \left|\left(Z_3 \times Z_3\right) : Z_4\right| \cdot \left|Z_3 \times Z_3\right| = 324$.

5) $G_{\alpha\beta} = Z_2 \times Z_2$

Because $A_4 = \left(Z_2 \times Z_2\right) : Z_3$, $S_4 = A_4 : Z_2$ and S_4 is the maximal subgroup of A_6, A_6 is the maximal subgroup of A_7, $N_H\left(Z_2 \times Z_2\right) = S_4$. And also because Z_3 does not centralize $Z_2 \times Z_2$, $C_H\left(Z_2 \times Z_2\right) = Z_2 \times Z_2$. So:

$\left|N_G\left(Z_2 \times Z_2\right)\right| = \left|C_H\left(Z_2 \times Z_2\right)\right| \cdot \left|N_H\left(Z_2 \times Z_2\right)\right| = \left|S_4\right| \cdot \left|Z_2 \times Z_2\right| = 96$.

6) $G_{\alpha\beta} = A_4$

Similar to $G_{\alpha\beta} = S_4$, by GAP, we have $N_H\left(A_4\right) = \left(A_4 \times Z_3\right) : Z_2$, $C_H\left(A_4\right) = Z_3$, so $\left|N_G\left(A_4\right)\right| = \left|C_H\left(A_4\right)\right| \cdot \left|N_H\left(A_4\right)\right| = \left|Z_3\right| \cdot \left|\left(A_4 \times Z_3\right) : Z_2\right| = 72$.

7) $G_{\alpha\beta} = Z_5$

Because A_5 is simple group, which has no normal subgroup, and the subgroups of A_5 is S_3, A_4, D_{10}, $N_H\left(Z_5\right) = D_{10}$, $C_H\left(Z_5\right) = Z_5$. So:

$\left|N_G\left(Z_5\right)\right| = \left|C_H\left(Z_5\right)\right| \cdot \left|N_H\left(Z_5\right)\right| =\left|\left|D_{10}\right| \cdot \left|Z_5\right| = 50$.

8) $G_{\alpha\beta} = Z_7$

$Z_7 : Z_3$ is a maximal subgroup of $L_3(2)$ and $L_3(2)$ is a maximal subgroup of A_7, so $N_H\left(Z_7\right) = Z_7 : Z_3$, $C_H\left(Z_7\right) = Z_7$. So:

$\left|N_G\left(Z_7\right)\right| = \left|C_H\left(Z_7\right)\right| \cdot \left|N_H\left(Z_7\right)\right| = \left|Z_7 : Z_3\right| \cdot \left|Z_7\right| = 147$.

In the following paragraph, we will discuss whether suborbits are self-paired or not.

1) For $l_1 = 1$, obviously, this suborbit is self-paired.

2) For $l_2 = 105$, $\left|N_G\left(G_{\alpha\beta}\right) : G_{\alpha\beta}\right| = \frac{72}{24} = 3$, so it is not self-paired.

3) For $l_3 = 70$, $\left|N_G\left(G_{\alpha\beta}\right) : G_{\alpha\beta}\right| = \frac{1296}{36} = 36$, so it is self-paired.

4) For $l_4 = 280$, $\left|N_G\left(G_{\alpha\beta}\right) : G_{\alpha\beta}\right| = \frac{324}{9} = 36$, so it is self-paired.

5) For $l_5 = 630$, $\left|N_G\left(G_{\alpha\beta}\right) : G_{\alpha\beta}\right| = \frac{96}{4} = 24$, so it is self-paired.

6) For $l_6 = 210$, $\left|N_G\left(G_{\alpha\beta}\right) : G_{\alpha\beta}\right| = \frac{72}{12} = 6$, so it is self-paired.

7) For $l_7 = 504$, $\left|N_G\left(G_{\alpha\beta}\right) : G_{\alpha\beta}\right| = \frac{50}{5} = 10$, so it is self-paired.

8) For $l_8 = 360$, $\left|N_G\left(G_{\alpha\beta}\right) : G_{\alpha\beta}\right| = \frac{147}{7} = 21$, so it is not self-paired.

By the discussion above and the relationship between the orbital of G on

$\Omega \times \Omega$ and the suborbit of G on Ω, we acquire the following result.

Proposition 3.3 Let $G = A_7 \times A_7$ acts primitively on $\Omega = \{Hx \mid x \in A_7 \times A_7\}$, where $H = \{(t,t) \mid t \in T\} \cong A_7$. Then the following results hold:

1) If the length of suborbit is 1, the corresponding orbital graph is a loop;

2) If the length of suborbit is 105, the corresponding orbital graph is not self-paired, so it is digraph and is also arc-transitive graph;

3) If the length of suborbit is 70, the corresponding orbital graph is self-paired, so it is an undirected arc-transitive graph;

4) If the length of suborbit is 280, the corresponding orbital graph is self-paired, so it is an undirected arc-transitive graph;

5) If the length of suborbit is 630, the corresponding orbital graph is self-paired, so it is an undirected arc-transitive graph;

6) If the length of suborbit is 210, the corresponding orbital graph is self-paired, so it is an undirected arc-transitive graph;

7) If the length of suborbit is 504, the corresponding orbital graph is self-paired, so it is an undirected arc-transitive graph;

8) If the length of suborbit is 360, the corresponding orbital graph is self-paired, so it is digraph and is also arc-transitive graph.

By the discussion above, we can acquire the result which is described in Theorem 1.1.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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