Applied Mathematics, 2022, 13, 87-100
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

The Ill-Posedness of Derivative Interpolation and Regularized Derivative Interpolation for Non-Bandlimited Functions

Weidong Chen

Department of Mathematics, 460 McBryde Hall, Virginia Tech, Blacksburg, VA, USA
Email: weidongc@vt.edu

How to cite this paper: Chen, W.D. (2022) The Ill-Posedness of Derivative Interpolation and Regularized Derivative Interpolation for Non-Bandlimited Functions. *Applied Mathematics*, **13**, 87-100.
https://doi.org/10.4236/am.2022.131008

Received: August 2, 2021
Accepted: January 23, 2022
Published: January 26, 2022

Abstract

In this paper, the ill-posedness of derivative interpolation is discussed, and a regularized derivative interpolation for non-bandlimited signals is presented. The convergence of the regularized derivative interpolation is studied. The numerical results are given and compared with derivative interpolation using the Tikhonov regularization method. The regularized derivative interpolation in this paper is more accurate in computation.

Keywords

Nonband-Limited Function, Derivative Interpolation, Ill-Posedness, Regularization

1. Introduction

The computation of the derivative is widely applied in science and engineering [1].

In this section, we present the problem of finding the derivative of non-band-limited signals by the sampling theorem.

Definition 1: Suppose a function $f \in L^2(\mathbf{R})$, its *Fourier transform* $\hat{f}$ is:

$$\hat{f}(\omega) = \mathbf{F}(f)(\omega) = \int_{-\infty}^{+\infty} f(t)\mathrm{e}^{-i\omega t}\mathrm{d}t,\ \omega \in \mathbf{R} \tag{1}$$

Definition 2: A function $f \in L^2(\mathbf{R})$ is said to be Ω *-band-limited* if $\hat{f}(\omega) = 0$, for every $\omega \notin [-\Omega, \Omega]$. Otherwise, it is non-bandlimited. Here $\hat{f}$ is the *Fourier transform* of f [2] [3].

The inversion formula is

$$\mathbf{F}^{-1}(\hat{f})(t) = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\mathrm{e}^{i\omega t}\mathrm{d}\omega,\ \text{a.e. } t \in \mathbf{R} \tag{2}$$

For band-limited signals, we have the Shannon sampling theorem [2] [3].

Shannon Sampling Theorem. If $f \in L^2(\mathbf{R})$ and is Ω *-band-limited*, then it can be exactly reconstructed from its samples $f(nh)$:

$$f(t) = \lim_{N\to\infty} \sum_{n=-N}^{N} SINC\Omega(t-nh) f(nh) \tag{3}$$

where $SINC\Omega(t-nh) := \dfrac{\sin\Omega(t-nh)}{\Omega(t-nh)}$ and $h = \pi/\Omega$. Here the convergence is in L^2 and uniformly on $\mathbf{R}$.

In [4], Marks presented an algorithm to find the derivative of band-limited signals by the sampling theorem:

$$f^{(k)}(t) = \sum_{n=-\infty}^{\infty} \left[SINC\Omega(t-nh)\right]^{(k)} f(nh). \tag{4}$$

Here, again, the convergence is in L^2 and uniformly on $\mathbf{R}$.

In [5], a method of numerical differentiation is given by low degree Chebyshev.

In this paper, we will consider the problem of computing $f^{(k)}$ from the samples of f in the presence of noise.

$$f(nh) = f_E(nh) + \eta(nh), \tag{5}$$

where $f_E(nh)$ is the exact signal and $\{\eta(nh)\}$ is the noise with the bound $\delta > 0$, $|\eta(nh)| \le \delta$.

Formula (4) is not reliable due to the ill-poseness. In [6], a regularized derivative interpolation formula is presented for Ω-bandlimited functions. In this paper, a regularized derivative interpolation formula will be presented for non-bandlimited functions. Its convergence property is proved and applications will be shown by some examples. In the case non-bandlimited functions, the error estimate is different and the step size h of the samples is necessary to be close to zero.

2. The Regularized Derivative Interpolation

In this section, we present the regularized derivative interpolation by the sampling theorem in the pair of spaces (C^k_{NB}, l^∞). Here

$$C^k_{NB} := \left\{ f^{(k)} : f \text{ is non-bandlimited} \right\}$$

with the norm of C^k_{NB} defined by

$$\left\| f^{(k)}(t) \right\|_{C^k_{NB}} := \max_{t\in\mathbf{R}} \left| f^{(k)}(t) \right| = \left\| f^{(k)}(t) \right\|_\infty,$$

and

$$l^\infty := \left\{ \{a(n) : n \in \mathbf{Z}\} : \|\mathbf{a}\|_{l^\infty} < \infty \right\}$$

is the space of bounded sequences with the norm

$$\|\mathbf{a}\|_{l^\infty} := \sup_{n\in\mathbf{Z}} |a(n)|.$$

We define the operator

$$\mathbf{S}: C_{NB}^{k} \to l^{\infty}, \text{ by } \mathbf{S}f^{(k)} := \left\{ f(nh): n \in \mathbf{Z} \right\}.$$

Here $f(nh)$ is the coefficient of $\left[SINC\Omega(t-nh)\right]^{(k)}$ in (4).

Remark 1. The problem of computing $f^{(k)}(t)$ from $f(nh)$ is an ill-posed problem.

To solve this ill-posed problem, we introduce the regularized Fourier transform [7] [8] [9] [10] [11]:

Definition 2. For $\alpha > 0$ we define

$$\mathbf{F}_{\alpha}[f](\omega) := \mathbf{F}[f_W](\omega) = \int_{-\infty}^{\infty} \frac{f(t)\mathrm{e}^{-i\omega t}\mathrm{d}t}{1+2\pi\alpha+2\pi\alpha t^2}. \tag{6}$$

where

$$f_W(t) := \frac{1}{1+2\pi\alpha+2\pi\alpha t^2} f(t)$$

is the function $f(t)$ multiplied with the weight function

$$K_{\alpha}(t) := \left(1+2\pi\alpha+2\pi\alpha t^2\right)^{-1}.$$

Definition 3. Given $\left\{ f(nh): n \in \mathbf{Z} \right\}$ in l^{∞}, define

$$f_{\alpha}(t) := \sum_{n=-\infty}^{\infty} SINC\Omega(t-nh) K_{\alpha}(nh) f(nh).$$

The infinite series is uniformly convergent in $\mathbf{R}$ for any $\alpha > 0$.

By the differentiation of $f_{\alpha}(t)$ in Definition 3, we obtain the regularized derivative interpolation:

$$\begin{aligned} f_{\alpha}'(t) &= \sum_{n=-\infty}^{\infty} \left[SINC\Omega(t-nh)\right]' K_{\alpha}(nh) f(nh) \\ &= \sum_{n=-\infty}^{\infty} \left[\frac{\cos\Omega(t-nh)}{t-nh} - \frac{\sin\Omega(t-nh)}{\Omega(t-nh)^2} \right] K_{\alpha}(nh) f(nh). \end{aligned} \tag{7}$$

This derivative is well defined since the infinite series is also uniformly convergent on $\mathbf{R}$.

Lemma 1. If f is non-band-limited and $\hat{f} \in L^1$, then

$$\mathbf{F}_{\alpha}\left[f(t)\right] = \frac{1}{4\pi a\alpha} \int_{-\infty}^{\infty} \hat{f}(u) \mathrm{e}^{-a|u-\omega|}\mathrm{d}u \quad \text{where} \quad a := \left(\frac{1+2\pi\alpha}{2\pi\alpha}\right)^{\frac{1}{2}}.$$

It can be seen from the convolution

$$\hat{f}_W = \frac{1}{2\pi} \hat{f} * \hat{K}_{\alpha}$$

where $\hat{K}_{\alpha}(\omega) = \frac{1}{2a\alpha}\mathrm{e}^{-\alpha|\omega|}$ is the Fourier transform of $K_{\alpha}(t)$. For the proof of the convergence of the regularized derivative interpolation we will need the definition of periodic extension of the function $\mathrm{e}^{i\omega t}$ [12].

Definition 4. $\left(\mathrm{e}^{i\omega t}\right)_{p[-\Omega,\Omega]}$ denotes the periodic extension of the function $\mathrm{e}^{i\omega t}$ defined on the interval $[-\Omega,\Omega]$ to the interval $(-\infty,\infty)$ with period

2Ω.

The next Lemma is from [12].

Lemma 2. If $\hat{f} \in L^1(-\infty,\infty)$, then

$$\sum_{n=-\infty}^{\infty} SINC\Omega(t-nh) f(nh) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\left(\mathrm{e}^{-i\omega t}\right)_{p[-\Omega,\Omega]} \mathrm{d}\omega$$

for each $t \in \mathbf{R}$.

Remark 2. If f is Ω-band-limited, Lemma 2 reduces to the Shannon sampling theorem.

Lemma 3. For bounded $\hat{f}$ on $[-\Omega,\Omega]$

$$\frac{1}{4\pi^2 a\alpha}\int_{|\omega|\ge\Omega}\left|\omega\int_{|u|\le\Omega}\hat{f}(u)\mathrm{e}^{-a|u-\omega|}\mathrm{d}u\right|\mathrm{d}\omega = O\left(\alpha^{1/2}\right).$$

The proof is in [6].

Lemma 4. Suppose $\Omega > 0$ and $a = \left(\frac{1+2\pi\alpha}{2\pi\alpha}\right)^{\frac{1}{2}}$ which is defined in lemma 1. For $u > \Omega$,

$$\int_{|\omega|\ge\Omega}|\omega|\,\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega = \frac{1}{a}\left[2u + \left(-\Omega+\frac{1}{a}\right)\mathrm{e}^{a(\Omega-u)}\right] + \frac{1}{a}\left(\Omega+\frac{1}{a}\right)\mathrm{e}^{-a(\Omega+u)}.$$

And if a is large enough, by omitting higher order infinitesimal we have

$$\int_{|\omega|\ge\Omega}|\omega|\,\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega \le \frac{2u}{a}.$$

Proof.

$$\int_{\omega\ge\Omega}\omega\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega = \int_{\Omega}^{u}\omega\mathrm{e}^{-a(u-\omega)}\mathrm{d}\omega + \int_{u}^{\infty}\omega\mathrm{e}^{a(u-\omega)}\mathrm{d}\omega$$
$$= \frac{1}{a}\left[2u + \left(-\Omega+\frac{1}{a}\right)\mathrm{e}^{a(\Omega-u)}\right].$$

$$\int_{\omega\le-\Omega}|\omega|\,\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega = \int_{\omega\le-\Omega}(-\omega)\mathrm{e}^{-a(u-\omega)}\mathrm{d}\omega = \frac{1}{a}\left(\Omega+\frac{1}{a}\right)\mathrm{e}^{-a(\Omega+u)}.$$

Lemma 5. Suppose $\Omega > 0$ and $a := \left(\frac{1+2\pi\alpha}{2\pi\alpha}\right)^{\frac{1}{2}}$. For $u < -\Omega$,

$$\int_{|\omega|\ge\Omega}|\omega|\,\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega = \frac{1}{a}\left[\left(-\Omega+\frac{1}{a}\right)\mathrm{e}^{a(\Omega+u)} - 2u\right] + \frac{1}{a}\left(\Omega+\frac{1}{a}\right)\mathrm{e}^{a(u-\Omega)}.$$

And if a is large enough we have

$$\int_{|\omega|\ge\Omega}|\omega|\,\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega \le \frac{-2u}{a}.$$

Proof.

$$\int_{\omega\ge\Omega}|\omega|\,\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega = \int_{\omega\ge\Omega}\omega\mathrm{e}^{-a(u-\omega)}\mathrm{d}\omega = \frac{1}{a}\left(\Omega+\frac{1}{a}\right)\mathrm{e}^{a(u-\Omega)}.$$

$$\int_{\omega\le-\Omega}|\omega|\,\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega = -\int_{-\infty}^{u}\omega\mathrm{e}^{-a(u-\omega)}\mathrm{d}\omega - \int_{u}^{-\Omega}\omega\mathrm{e}^{a(u-\omega)}\mathrm{d}\omega$$
$$= \frac{1}{a}\left[\left(-\Omega+\frac{1}{a}\right)\mathrm{e}^{a(\Omega+u)} - 2u\right].$$

Lemma 6. If $u\hat{f}(u)\in L^1(R)$ then

$$\frac{1}{4\pi^2 a\alpha}\int_{|\omega|\ge\Omega}\left|\omega\int_{|u|\ge\Omega}\hat{f}(u)\mathrm{e}^{-a|u-\omega|}\mathrm{d}u\right|\mathrm{d}\omega\le\frac{1}{2\pi(1+2\pi\alpha)}\int_{|u|\ge\Omega}\left|u\hat{f}(u)\right|\mathrm{d}u.$$

Proof. By lemma 4 and 5

$$\begin{aligned}&\int_{|\omega|\ge\Omega}\left|\omega\int_{|u|\ge\Omega}\hat{f}(u)\mathrm{e}^{-a|u-\omega|}\mathrm{d}u\right|\mathrm{d}\omega\le\int_{|\omega|\ge\Omega}|\omega|\int_{|u|\ge\Omega}\left|\hat{f}(u)\right|\mathrm{e}^{-a|u-\omega|}\mathrm{d}u\mathrm{d}\omega\\&=\int_{u\ge\Omega}\left|\hat{f}(u)\right|\mathrm{d}u\int_{|\omega|\ge\Omega}|\omega|\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega+\int_{u\le-\Omega}\left|\hat{f}(u)\right|\mathrm{d}u\int_{|\omega|\ge\Omega}|\omega|\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega\\&\le\int_{u\ge\Omega}\left|\hat{f}(u)\right|\frac{2u}{a}\mathrm{d}u+\int_{u\le-\Omega}\left|\hat{f}(u)\right|\frac{-2u}{a}\mathrm{d}u=\frac{2}{a}\int_{|u|\ge\Omega}\left|u\hat{f}(u)\right|\mathrm{d}u.\end{aligned}$$

So

$$\frac{1}{4\pi^2 a\alpha}\int_{|\omega|\ge\Omega}\left|\omega\int_{|u|\ge\Omega}\hat{f}(u)\mathrm{e}^{-a|u-\omega|}\mathrm{d}u\right|\mathrm{d}\omega\le\frac{1}{2\pi(1+2\pi\alpha)}\int_{|u|\ge\Omega}\left|u\hat{f}(u)\right|\mathrm{d}u.$$

Lemma 7. Assume $\omega\hat{f}(\omega)\in L^1(\mathbf{R})$ and $\hat{f}\in L^\infty[-\Omega,\Omega]$. As $\alpha\to 0$, we have

$$\frac{1}{\pi}\int_{|\omega|\ge\Omega}\left|\omega\hat{f}_W(\omega)\right|\mathrm{d}\omega=O\left(\alpha^{1/2}\right)+\frac{1}{2\pi(1+2\pi\alpha)}\int_{|\omega|\ge\Omega}\left|\omega\hat{f}(\omega)\right|\mathrm{d}\omega$$

Proof.

$$\begin{aligned}\frac{1}{\pi}\int_{|\omega|\ge\Omega}\left|\omega\hat{f}_W(\omega)\right|\mathrm{d}\omega&=\frac{1}{\pi}\int_{|\omega|\ge\Omega}\left|\omega\frac{1}{4\pi a\alpha}\int_{-\infty}^{\infty}\hat{f}(u)\mathrm{e}^{-a|u-\omega|}\mathrm{d}u\right|\mathrm{d}\omega\\&=\frac{1}{4\pi^2 a\alpha}\int_{|\omega|\ge\Omega}\left|\omega\int_{-\infty}^{\infty}\hat{f}(u)\mathrm{e}^{-a|u-\omega|}\mathrm{d}u\right|\mathrm{d}\omega\\&\le\frac{1}{4\pi^2 a\alpha}\int_{|\omega|\ge\Omega}\left|\omega\int_{|u|\le\Omega}\hat{f}(u)\mathrm{e}^{-a|u-\omega|}\mathrm{d}u\right|\mathrm{d}\omega\\&\quad+\frac{1}{4\pi^2 a\alpha}\int_{|\omega|\ge\Omega}\left|\omega\int_{|u|\ge\Omega}\hat{f}(u)\mathrm{e}^{-a|u-\omega|}\mathrm{d}u\right|\mathrm{d}\omega.\end{aligned}$$

By lemma 3

$$\frac{1}{4\pi^2 a\alpha}\int_{|\omega|\ge\Omega}\left|\omega\int_{|u|\le\Omega}\hat{f}(u)\mathrm{e}^{-a|u-\omega|}\mathrm{d}u\right|\mathrm{d}\omega=O\left(\alpha^{1/2}\right).$$

By lemma 6

$$\frac{1}{4\pi^2 a\alpha}\int_{|\omega|\ge\Omega}\left|\omega\int_{|u|\ge\Omega}\hat{f}(u)\mathrm{e}^{-a|u-\omega|}\mathrm{d}u\right|\mathrm{d}\omega\le\frac{1}{2\pi(1+2\pi\alpha)}\int_{|u|\ge\Omega}\left|u\hat{f}(u)\right|\mathrm{d}u.$$

In order to prove the convergence property of the regularized derivative interpolation we will need the next lemma.

Lemma 8. If f is non-band-limited, $\hat{f}\in L^\infty[-\Omega,\Omega]$ and $\omega\hat{f}(\omega)\in L^1(\mathbf{R})$, then

$$\begin{aligned}d&:=\left|\sum_{n=-\infty}^{\infty}\left[SINC\Omega(t-nh)\right]'K_\alpha(nh)f(nh)-\left[K_\alpha(t)f(t)\right]'\right|\\&=O\left(\alpha^{\frac{1}{2}}\right)+\frac{1}{2\pi(1+2\pi\alpha)}\int_{|\omega|\ge\Omega}\left|\omega\hat{f}(\omega)\right|\mathrm{d}\omega.\end{aligned}$$

Proof.

$$
\begin{aligned}
d &= \left| \left[\sum_{n=-\infty}^{\infty} SINC\Omega(t-nh) K_\alpha(nh) f(nh) - K_\alpha(t) f(t) \right]' \right| \\
&= \left| \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\mathrm{e}^{-i\omega t}\right)_{p[-\Omega,\Omega]} \hat{f}_W(\omega)\mathrm{d}\omega - K_\alpha(t) f(t) \right]' \right| \\
&= \left| \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\mathrm{e}^{-i\omega t}\right)_{p[-\Omega,\Omega]} \hat{f}_W(\omega)\mathrm{d}\omega - \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-i\omega t} \hat{f}_W(\omega)\mathrm{d}\omega \right]' \right| \\
&= \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\left(\mathrm{e}^{-i\omega t}\right)_{p[-\Omega,\Omega]} - \mathrm{e}^{-i\omega t} \right]' \hat{f}_W(\omega)\mathrm{d}\omega \right| \\
&= \left| \frac{1}{2\pi} \int_{|\omega|\ge\Omega} \left[\left(i\omega\mathrm{e}^{-i\omega t}\right)_{p[-\Omega,\Omega]} - i\omega\mathrm{e}^{-i\omega t} \right] \hat{f}_W(\omega)\mathrm{d}\omega \right| \\
&\le \frac{1}{\pi} \int_{|\omega|\ge\Omega} \left| \omega \hat{f}_W(\omega) \right| \mathrm{d}\omega.
\end{aligned}
$$

Then by Lemma 7, we can see the estimate is true.

Lemma 9. If we choose $\alpha = \alpha(\delta)$ such that $\alpha \to 0$ and $\delta/\sqrt{\alpha} \to 0$ as $\delta \to 0$, then

$$
\left| \sum_{n=-\infty}^{\infty} \left[SINC\Omega(t-nh) \right]' K_\alpha(nh)\eta(nh) \right| = O(\delta) + O\left(\delta/\sqrt{\alpha}\right).
$$

The proof is in [6].

We are now in a position to state and prove our main theorem.

Theorem 1. Suppose $f(nh) = f_E(nh) + \eta(nh)$ where $\|\eta\|_\infty \le \delta$ and $f_E \in L^2$ is non-band-limited, $\hat{f}_E \in L^\infty[-\Omega,\Omega]$ and $\omega \hat{f}_E(\omega) \in L^1(\mathbf{R})$. Then if we choose $\alpha = \alpha(\delta)$ such that $\alpha(\delta) \to 0$ and $\delta/\sqrt{\alpha(\delta)} \to 0$ as $\delta \to 0$, then

$$
\left\| f_\alpha'(t) - f_E'(t) \right\|_{C[-T,T]} \le O\left(\alpha^{\frac{1}{2}}\right) + O\left(\delta/\sqrt{\alpha}\right) + \frac{1}{2\pi(1+2\pi\alpha)} \int_{|\omega|\ge\Omega} \left| \omega \hat{f}_E(\omega) \right| \mathrm{d}\omega.
$$

Proof. Suppose $t \in [-T,T]$. Using Formula (7) and Lemma 8, we obtain

$$
\begin{aligned}
f_\alpha'(t) - f_E'(t) &= \sum_{n=-\infty}^{\infty} \left[SINC\Omega(t-nh) \right]' K_\alpha(nh)\left[f_E(nh) + \eta(nh) \right] - f_E'(t) \\
&= \sum_{n=-\infty}^{\infty} \left[SINC\Omega(t-nh) \right]' K_\alpha(nh) f_E(nh) - f_E'(t) \\
&\quad + \sum_{n=-\infty}^{\infty} \left[SINC\Omega(t-nh) \right]' K_\alpha(nh)\eta(nh) \\
&= \sum_{n=-\infty}^{\infty} \left[SINC\Omega(t-nh) \right]' K_\alpha(nh) f_E(nh) - \left[K_\alpha(t) f_E(t) \right]' \\
&\quad + \left[K_\alpha(t) f_E(t) \right]' - f_E'(t) + \sum_{n=-\infty}^{\infty} \left[SINC\Omega(t-nh) \right]' K_\alpha(nh)\eta(nh) \\
&= O\left(\alpha^{\frac{1}{2}}\right) + \frac{1}{2\pi(1+2\pi\alpha)} \int_{|\omega|\ge\Omega} \left| \omega \hat{f}_E(\omega) \right| \mathrm{d}\omega \\
&\quad - \frac{\left(2\pi\alpha + 2\pi\alpha t^2\right)\left(1 + 2\pi\alpha + 2\pi\alpha t^2\right) f_E'(t) + 4\pi\alpha f_E(t)}{\left(1 + 2\pi\alpha + 2\pi\alpha t^2\right)^2}
\end{aligned}
$$

$$+\sum_{n=-\infty}^{\infty}\left[SINC\Omega(t-nh)\right]' K_\alpha(nh)\eta(nh)$$
$$=O\left(\alpha^{\frac{1}{2}}\right)+\frac{1}{2\pi(1+2\pi\alpha)}\int_{|\omega|\geq\Omega}\left|\omega\hat{f}_E(\omega)\right|\mathrm{d}\omega$$
$$+O(\alpha)+\sum_{n=-\infty}^{\infty}\left[SINC\Omega(t-nh)\right]' K_\alpha(nh)\eta(nh).$$

By Lemma 9, we have

$$\left\|f_\alpha'(t)-f_E'(t)\right\|_{C[-T,T]}\leq O\left(\alpha^{\frac{1}{2}}\right)+O\left(\delta/\sqrt{\alpha}\right)+\frac{1}{2\pi(1+2\pi\alpha)}\int_{|\omega|\geq\Omega}\left|\omega\hat{f}_E(\omega)\right|\mathrm{d}\omega.$$

Remark 3. According this theorem, if we choose Ω to be large enough, and $\alpha=\alpha(\delta)$ such that $\alpha(\delta)\to 0$ and $\delta/\sqrt{\alpha(\delta)}\to 0$ as $\delta\to 0$, we can have good approximation.

3. Derivative Interpolation of Higher Order

In this section, we prove the convergence property of the derivative interpolation formula of high order:

$$f_\alpha^{(k)}(t)=\sum_{n=-\infty}^{\infty}\left[SINC\Omega(t-nh)\right]^{(k)} K_\alpha(nh)f(nh). \tag{8}$$

Lemma 10.

$$I_k:=\int_\Omega^u\omega^k\mathrm{e}^{a\omega}\mathrm{d}\omega=\sum_{l=0}^{k}\frac{(-1)^l A_k^l}{a^{l+1}}\left(u^{k-l}\mathrm{e}^{au}-\Omega^{k-l}\mathrm{e}^{a\Omega}\right)$$

where $A_k^l:=\prod_{j=1}^{l}(k-j+1)$ and $A_k^0:=1$.

Proof.

$$I_k=\frac{1}{a}\left(u^k\mathrm{e}^{au}-\Omega^k\mathrm{e}^{-a\Omega}\right)-\frac{k}{a}I_{k-1}=\cdots=\sum_{l=0}^{k}\frac{(-1)^l A_k^l}{a^{l+1}}\left(u^{k-l}\mathrm{e}^{au}-\Omega^{k-l}\mathrm{e}^{a\Omega}\right).$$

Lemma 11.

$$J_k(\Omega):=\int_{\omega\geq\Omega}\omega^k\mathrm{e}^{-a\omega}\mathrm{d}\omega=\sum_{l=0}^{k}\frac{A_k^l}{a^{l+1}}\Omega^{k-l}\mathrm{e}^{-a\Omega}.$$

The proof is similar to the proof of Lemma 10.

Lemma 12. If $u>\Omega$, then

$$\int_{\omega\geq\Omega}|\omega|^k\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega=\mathrm{e}^{-au}I_k+\mathrm{e}^{au}J_k(u)\leq Cu^k.$$

and

$$\int_{\omega\leq-\Omega}|\omega|^k\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega=\mathrm{e}^{-au}J_k(\Omega).$$

Proof.

$$\int_{\omega\geq\Omega}|\omega|^k\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega=\int_\Omega^u\omega^k\mathrm{e}^{-a(u-\omega)}\mathrm{d}\omega+\int_u^\infty\omega^k\mathrm{e}^{-a(\omega-u)}\mathrm{d}\omega$$
$$=\mathrm{e}^{-au}\int_\Omega^u\omega^k\mathrm{e}^{a\omega}\mathrm{d}\omega+\mathrm{e}^{au}\int_u^\infty\omega^k\mathrm{e}^{-a\omega}\mathrm{d}\omega=\mathrm{e}^{-au}I_k+\mathrm{e}^{au}J_k(u)\leq Cu^k.$$

$$\int_{\omega\leq-\Omega}|\omega|^k\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega=(-1)^k\int_{\omega\leq-\Omega}\omega^k\mathrm{e}^{-a(u-\omega)}\mathrm{d}\omega$$
$$=(-1)^k\mathrm{e}^{-au}\int_{\omega\leq-\Omega}\omega^k\mathrm{e}^{a\omega}\mathrm{d}\omega=\mathrm{e}^{-au}\int_{\omega_1\geq\Omega}\omega_1^k\mathrm{e}^{-a\omega_1}\mathrm{d}\omega_1=\mathrm{e}^{-au}J_k(\Omega)$$

where $\omega_1 = -\omega$.

Lemma 13. If $\left|\hat{f}\right| \le M$ on $[-\Omega, \Omega]$ and $\alpha \to 0$

$$\frac{1}{4\pi^2 a\alpha}\int_{|\omega|\ge\Omega}\left|\omega^k \int_{|u|\le\Omega}\hat{f}(u)\mathrm{e}^{-a|u-\omega|}\mathrm{d}u\right|\mathrm{d}\omega = O\left(\alpha^{1/2}\right).$$

Proof.

$$LHS \le 2\int_{\omega\ge\Omega}\omega^k \frac{M}{4\pi^2 a^2\alpha}\mathrm{e}^{-a\omega}\mathrm{e}^{a\Omega}\mathrm{d}\omega = \frac{\mathrm{e}^{a\Omega}}{2\pi^2 a^2\alpha}\int_{\omega\ge\Omega}\omega^k\mathrm{e}^{-a\omega}\mathrm{d}\omega$$

$$= \frac{\mathrm{e}^{a\Omega}}{2\pi^2 a^2\alpha}\sum_{l=0}^{k}\frac{A_k^l}{a^{l+1}}\Omega^{k-l}\mathrm{e}^{-a\Omega} = \frac{1}{2\pi^2 a^2\alpha}\sum_{l=0}^{k}\frac{A_k^l}{a^{l+1}}\Omega^{k-l} = O\left(\alpha^{1/2}\right),$$

Lemma 14. If $\omega^k\hat{f}(\omega)\in L^1(\mathbf{R})$, then

$$\frac{1}{4\pi^2 a\alpha}\int_{|\omega|\ge\Omega}\left|\omega^k\int_{|u|\ge\Omega}\hat{f}(u)\mathrm{e}^{-a|u-\omega|}\mathrm{d}u\right|\mathrm{d}\omega \le C\int_{|u|\ge\Omega}\left|u^k\hat{f}(u)\right|\mathrm{d}u$$

where $C = const. > 0$.

Proof.

$$LHS \le \frac{1}{4\pi^2 a\alpha}\int_{|u|\ge\Omega}\left|\hat{f}(u)\right|\mathrm{d}u\int_{|\omega|\ge\Omega}|\omega|^k\,\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega = M_1 + M_2$$

$$M_1 = \frac{1}{4\pi^2 a\alpha}\int_{u\ge\Omega}\left|\hat{f}(u)\right|\mathrm{d}u\int_{\omega\ge\Omega}|\omega|^k\,\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega + \frac{1}{4\pi^2 a\alpha}\int_{u\ge\Omega}\left|\hat{f}(u)\right|\mathrm{d}u\int_{\omega\le-\Omega}|\omega|^k\,\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega$$

$$M_2 = \frac{1}{4\pi^2 a\alpha}\int_{u\le-\Omega}\left|\hat{f}(u)\right|\mathrm{d}u\int_{\omega\ge\Omega}|\omega|^k\,\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega + \frac{1}{4\pi^2 a\alpha}\int_{u\le-\Omega}\left|\hat{f}(u)\right|\mathrm{d}u\int_{\omega\le-\Omega}|\omega|^k\,\mathrm{e}^{-a|u-\omega|}\mathrm{d}\omega.$$

By Lemma 12, $M_1 \le C\int_{u\ge\Omega}\left|u^k\hat{f}(u)\right|\mathrm{d}u$. Similarly, $M_2 \le C\int_{u\le-\Omega}\left|u^k\hat{f}(u)\right|\mathrm{d}u$.

Lemma 15. If $\hat{f}\in L^\infty[-\Omega,\Omega]$, $\omega^k\hat{f}(\omega)\in L^1(\mathbf{R})$ and $\alpha\to 0$, then

$$\frac{1}{\pi}\int_{|\omega|\ge\Omega}\left|\omega^k\hat{f}_W(\omega)\right|\mathrm{d}\omega \le O\left(\alpha^{\frac{1}{2}}\right) + C\int_{|\omega|\ge\Omega}\left|\omega^k\hat{f}(\omega)\right|\mathrm{d}\omega$$

where $C = const. > 0$

Proof. By Lemma 13 and 14

$$\int_{|\omega|\ge\Omega}\left|\omega^k\hat{f}_W(\omega)\right|\mathrm{d}\omega \le \frac{1}{4\pi^2 a\alpha}\int_{|\omega|\ge\Omega}\left|\omega^k\int_{|u|\le\Omega}\hat{f}(u)\mathrm{e}^{-a|u-\omega|}\mathrm{d}u\right|\mathrm{d}\omega + \frac{1}{4\pi^2 a\alpha}\int_{|\omega|\ge\Omega}\left|\omega^k\int_{|u|\ge\Omega}\hat{f}(u)\mathrm{e}^{-a|u-\omega|}\mathrm{d}u\right|\mathrm{d}\omega$$

$$\le O\left(\alpha^{\frac{1}{2}}\right) + C\int_{|\omega|\ge\Omega}\left|\omega^k\hat{f}(\omega)\right|\mathrm{d}\omega$$

Lemma 16. If f is non-band-limited, $\hat{f}\in L^\infty[-\Omega,\Omega]$, $\omega^k\hat{f}(\omega)\in L^1(\mathbf{R})$ and $\alpha\to 0$ then

$$d := \left|\sum_{n=-\infty}^{\infty}\left[SINC\Omega(t-nh)\right]^{(k)}K_\alpha(nh)f(nh) - \left[K_\alpha(t)f(t)\right]^{(k)}\right| = O\left(\alpha^{\frac{1}{2}}\right) + C\int_{|\omega|\ge\Omega}\left|\omega^k\hat{f}(\omega)\right|\mathrm{d}\omega$$

where $C = const. > 0$.

Proof. Since

$$\sum_{n=-\infty}^{\infty}\left[SINC\Omega(t-nh)\right]^{(k)} K_\alpha(nh) f(nh)$$

is uniformly convergent,

$$\begin{aligned}
d &= \left|\left[\sum_{n=-\infty}^{\infty} SINC\Omega(t-nh) K_\alpha(nh) f(nh) - K_\alpha(t) f(t)\right]^{(k)}\right| \\
&= \left|\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{i\omega t}\right)_{p[-\Omega,\Omega]} \hat{f}_W(\omega)\mathrm{d}\omega - K_\alpha(t) f(t)\right]^{(k)}\right| \\
&= \left|\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{i\omega t}\right)_{p[-\Omega,\Omega]} \hat{f}_W(\omega)\mathrm{d}\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\hat{f}_W(\omega)\mathrm{d}\omega\right]^{(k)}\right| \\
&= \left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\left(e^{i\omega t}\right)_{p[-\Omega,\Omega]} - e^{i\omega t}\right]^{(k)} \hat{f}_W(\omega)\mathrm{d}\omega\right| \\
&= \left|\frac{1}{2\pi}\int_{|\omega|\ge\Omega}\left[(i\omega)^k\left(e^{i\omega t}\right)_{p[-\Omega,\Omega]} - (i\omega)^k e^{i\omega t}\right]\hat{f}_W(\omega)\mathrm{d}\omega\right| \\
&\le \frac{2}{2\pi}\int_{|\omega|\ge\Omega}\left|\omega^k \hat{f}_W(\omega)\right|\mathrm{d}\omega = O\left(\alpha^{1/2}\right) + C\int_{|\omega|\ge\Omega}\left|\omega^k \hat{f}(\omega)\right|\mathrm{d}\omega.
\end{aligned}$$

Lemma 17. For $t \in [-T, T]$, if k is even

$$\left[K_\alpha(t)\right]^{(k)} = \frac{1}{2\pi\alpha}\left[\frac{1}{a^2+t^2}\right]^{(k)} = O\left(\alpha^{k/2}\right),$$

and if k is odd

$$\left[K_\alpha(t)\right]^{(k)} = \frac{1}{2\pi\alpha}\left[\frac{1}{a^2+t^2}\right]^{(k)} = O\left(\alpha^{(k+1)/2}\right),$$

as $\alpha \to 0$.

Proof is in [6].

Lemma 18. For $t \in [-T, T]$,

$$\left[K_\alpha(t) f_E(t)\right]^{(k)} - f_E^{(k)}(t) = O(\alpha).$$

Proof is in [6].

Now we can state and prove a version of Theorem 1 for higher order derivatives.

Theorem 2. Assume $\omega^k \hat{f}(\omega) \in L^1(\mathbf{R})$ and $\hat{f} \in L^\infty[-\Omega, \Omega]$. If we choose $\alpha = \alpha(\delta)$ such that $\alpha(\delta) \to 0$ and $\delta/\sqrt{\alpha(\delta)} \to 0$ as $\delta \to 0$, then we have the estimate

$$\left\| f_\alpha^{(k)}(t) - f_E^{(k)}(t)\right\|_{C[-T,T]} \le O\left(\alpha^{\frac{1}{2}}\right) + O\left(\delta/\sqrt{\alpha}\right) + C\int_{|\omega|\ge\Omega}\left|\omega^k \hat{f}_E(\omega)\right|\mathrm{d}\omega.$$

Proof.

$$
\begin{aligned}
& f_\alpha^{(k)}(t)-f_E^{(k)}(t) \\
& =\sum_{n=-\infty}^{\infty}\left[SINC\Omega(t-nh)\right]^{(k)} K_\alpha(nh)\left[f_E(nh)+\eta(nh)\right]-f_E^{(k)}(t) \\
& =\sum_{n=-\infty}^{\infty}\left[SINC\Omega(t-nh)\right]^{(k)} K_\alpha(nh) f_E(nh)-f_E^{(k)}(t) \\
& \quad+\sum_{n=-\infty}^{\infty}\left[SINC\Omega(t-nh)\right]^{(k)} K_\alpha(nh)\eta(nh) \\
& =\sum_{n=-\infty}^{\infty}\left[SINC\Omega(t-nh)\right]^{(k)} K_\alpha(nh) f_E(nh)-\left[K_\alpha(t) f_E(t)\right]^{(k)} \\
& \quad+\left[K_\alpha(t) f_E(t)\right]^{(k)}-f_E^{(k)}(t)+\sum_{n=-\infty}^{\infty}\left[SINC\Omega(t-nh)\right]^{(k)} K_\alpha(nh)\eta(nh)
\end{aligned}
$$

where

$$
\begin{aligned}
& \left|\sum_{n=-\infty}^{\infty}\left[SINC\Omega(t-nh)\right]^{(k)} K_\alpha(nh)\eta(nh)\right| \\
& \leq\left|\left[SINC\Omega t\right]^{(k)} \frac{\eta(0)}{1+2\pi\alpha}\right|+\left|\sum_{n\neq 0}\left[SINC\Omega(t-nh)\right]^{(k)} K_\alpha(nh)\eta(nh)\right| \\
& =O(\delta)+O\left(\delta\int_0^{\infty} K_\alpha(t)\,\mathrm{d}x\right)=O(\delta)+O\left(\delta/\sqrt{\alpha}\right).
\end{aligned}
$$

This implies

$$
\left\|f_\alpha^{(k)}(t)-f_E^{(k)}(t)\right\|_{C[-T,T]} \leq O\left(\alpha^{\frac{1}{2}}\right)+O\left(\delta/\sqrt{\alpha}\right)+C\int_{|\omega|\geq\Omega}\left|\omega^k \hat{f}_E(\omega)\right|\mathrm{d}\omega.
$$

Remark 4. We will choose the regularization parameter by the experiment. According to Theorem 2, α depends on δ. If we choose $\alpha=k\delta^\mu$, $k>0$, $0<\mu<2$, the assumptions of Theorem 2 are satisfied. If δ is known, α can be determined by discrepancy principle ([13]). The GCV and L-curve can be used ([14] [15]) if δ is not known.

4. Experimental Results

In this section, we give some examples to compare the regularized derivative interpolation by sampling with the Tikhonov regularization method [16] [17].

In practice, only finite terms can be used in (8). So we choose a large integer N, and use next formula in computation:

$$
f_\alpha^{(k)}(t)=\sum_{n=-N}^{N}\left[SINC\Omega(t-nh)\right]^{(k)} K_\alpha(nh) f(nh) \tag{9}
$$

where $f(nh)$ is the noisy sampling data given in (4) in the section of introduction. Due to the weight function, the series above converges much faster than the series (3) of using Shannon's sampling theorem. We give the estimate of the truncation error next

$$
\mathrm{TR}=O\left(\frac{1}{N^2\alpha}\right).
$$

So if N is large enough, the truncation error can be very small.

Suppose $f_T(t)=\dfrac{1}{1+t^2}$.

Then

$$\hat{f}_T(\omega) = \pi e^{-|\omega|}.$$

So $f_T(t)$ is not a band-limited signal.

In examples 1 and 2 we consider $f = f_E + \eta$.

Example 1. We choose the noise

$$\eta(nh) = \delta * sgn\{\cos\Omega(t_0 - nh)/\Omega(t_0 - nh)\}$$

where $\Omega = 1.5$, $t_0 = 0$, and the signals $f = f_E + \eta$ with $\delta = 0.001$. We choose $\alpha = 0.05$. The results of $f'(t)$ and $f''(t)$ are in **Figure 1** and **Figure 2**.

Example 2. We choose the noise to be white noise that is uniformly distributed in $[-0.005, 0.005]$. We choose $\alpha = 0.05$. The results of $f'(t)$ and $f''(t)$ are in **Figure 3** and **Figure 4**.

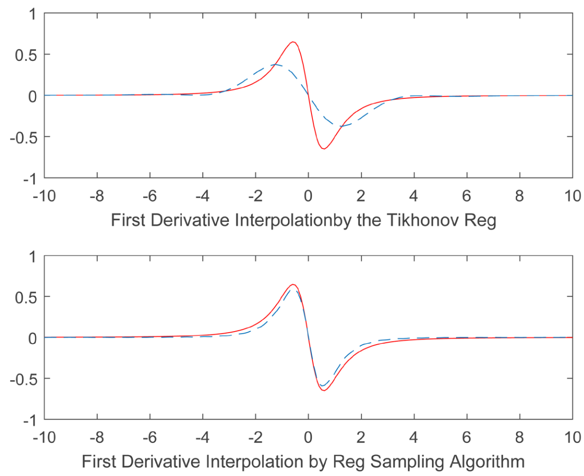

Figure 1. The result of Example 1. The solid curve is $f_E'(t)$. The dashed is the reg derivative $f_\alpha'(t)$.

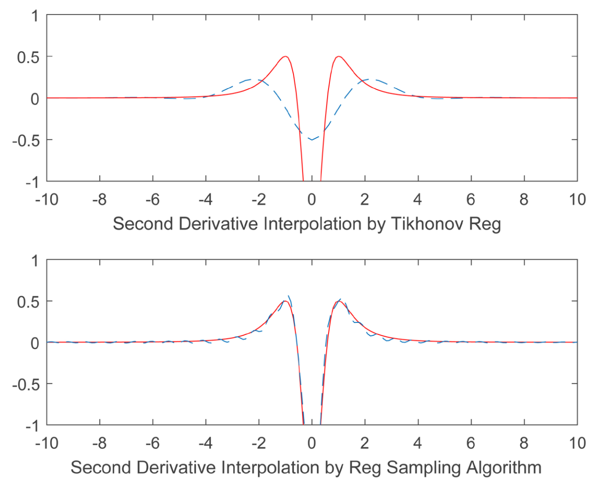

Figure 2. The result of Example 1. The solid curve is $f_E''(t)$. The dashed is the reg derivative $f_\alpha''(t)$.

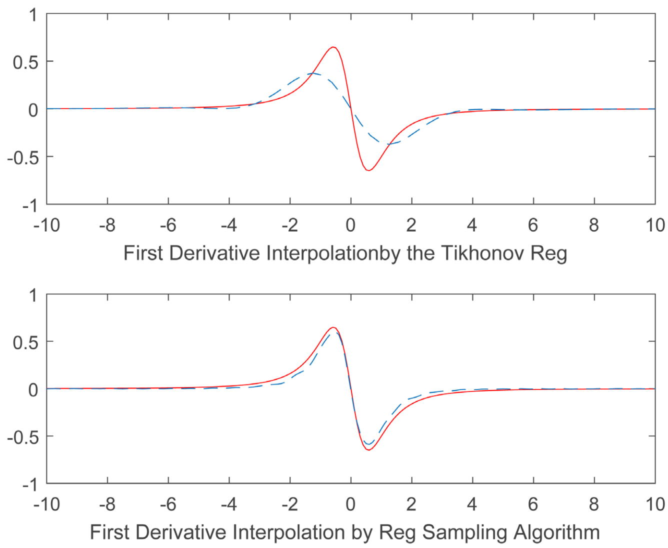

Figure 3. The result of Example 2. The solid curve is $f_E'(t)$. The dashed is the reg derivative $f_\alpha'(t)$.

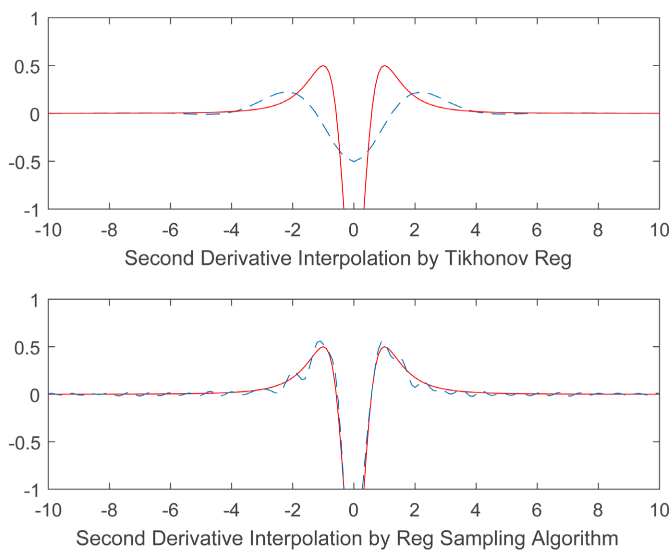

Figure 4. The result of Example 2. The solid curve is $f_E''(t)$. The dashed is the reg derivative $f_\alpha''(t)$.

5. Conclusion

The computation of derivatives is a highly ill-posed problem. The regularized derivative interpolation by sampling can be applied. The convergence property is proved and tested by some examples. The numerical results are better than Tikhonov regularization method.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Zhao, X.R., Sheng, Z., Li, J.W., Yu, H. and Wei, K.J. (2019) Determination of the "Wave Turbopause" Using a Numerical Differentiation Method. *Journal of Geophysical Research*: *Atmospheres*, **124**, 10592-10607. https://doi.org/10.1029/2019JD030754

[2] Shannon, C.E. (1948) A Mathematical Theory of Communication. *The Bell System Technical Journal*, **27**, 623-656. https://doi.org/10.1002/j.1538-7305.1948.tb00917.x

[3] Steiner, A. (1980) Plancherel's Theorem and the Shannon Series Derived Simultaneously. *The American Mathematical Monthly*, **87**, 193-197. https://doi.org/10.1080/00029890.1980.11994990

[4] Marks, R. (1983) Noise Sensitivity of Band-Limited Signal Derivative Interpolation. *IEEE Transactions on Acoustics*, *Speech*, *and Signal Processing*, **31**, 1028-1032. https://doi.org/10.1109/TASSP.1983.1164168

[5] Bruno, O. and Hoch, D. (2012) Numerical Differentiation of Approximated Functions with Limited Order-of-Accuracy Deterioration. *SIAM Journal on Numerical Analysis*, **50**, 1581-1603. https://doi.org/10.1137/100805807

[6] Chen, W.D. (2020) The Ill-Posedness of Derivative Interpolation and Regularized Derivative Interpolation for Band-Limited Functions. *EURASIP Journal on Advances in Signal Processing*, **2020**, Article No. 32. https://doi.org/10.1186/s13634-020-00668-5

[7] Chen, W.D. (2006) An Efficient Method for An Ill-Posed Problem—Band-Limited Extrapolation by Regularization. *IEEE Transactions on Signal Processing*, **54**, 4611-4618. https://doi.org/10.1109/TSP.2006.881255

[8] Chen, W.D. (2011) Computation of Fourier Transforms for Noisy Bandlimited Signals. *SIAM Journal on Numerical Analysis*, **49**, 1-14. https://doi.org/10.1137/100784758

[9] Chen, W.D. (2016) A Regularized Sampling Algorithm in Reconstructing Non-Bandlimited Functions. *Journal of Computational and Applied Mathematics*, **301**, 259-270. https://doi.org/10.1016/j.cam.2015.11.032

[10] Chen, W.D. (2011) The Ill-Posedness of the Sampling Problem and Regularized Sampling Algorithm. *Digital Signal Processing*, **21**, 375-390. https://doi.org/10.1016/j.dsp.2010.06.003

[11] Chen, W.D. (2017) A Regularized Two-Dimensional Sampling Algorithm. *Journal of Inverse and Ill-Posed Problems*, **26**, 67-84. https://doi.org/10.1515/jiip-2015-0049

[12] Brown, J.L. (1967) On the Error in Reconstructing a Non-Bandlimited Function by Means of the Bandpass Sampling Theorem. *Journal of Mathematical Analysis and Applications*, **18**, 75-84. https://doi.org/10.1016/0022-247X(67)90183-7

[13] Griesbaum, A., Barbara, B. and Vexler, B. (2008) Efficient Computation of the Tikhonov Regularization Parameter by Goal-Oriented Adaptive Discretization. *Inverse Problems*, **24**, Article ID: 025025. https://doi.org/10.1088/0266-5611/24/2/025025

[14] Belge, M., Kilmer, M.E. and Miller, E.L. (2002) Efficient Determination of Multiple Regularization Parameters in a Generalized L-Curve Framework. *Inverse Problems*, **18**, 1161-1183. https://doi.org/10.1088/0266-5611/18/4/314

[15] Kilmer, M.E. and O'leary, D.P. (2001) Choosing Regularization Parameters in Iterative Methods for Ill-Posed Problems. *SIAM Journal on Matrix Analysis and Applications*, **22**, 1204-1221. https://doi.org/10.1137/S0895479899345960

[16] Tikhonov, A.N. and Arsenin, V.Y. (1977) Solution of Ill-Posed Problems. Winston/Wiley, Washington DC.

[17] Wang, Y.B., Hon, Y.C. and Cheng, J. (2006) Reconstruction of High Order Derivatives from Input Data. *Journal of Inverse and Ill-Posed Problems*, **14**, 205-218. https://doi.org/10.1515/156939406777571085