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The Family of Exponential Attractors and Inertial Manifolds for a Generalized Nonlinear Kirchhoff Equations

Guoguang Lin*, **Xiaomei Liu**

Department of Mathematics, Yunnan University, Kunming, China
Email: *gglin@ynu.edu.cn

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Abstract

In this paper, we study the long-time behavior of a class of generalized nonlinear Kichhoff equation under the condition of n dimension. Firstly, the Lipschitz property and squeezing property of the nonlinear semigroup related to the initial-boundary value problem are proved, and then the existence of its exponential attractor is obtained. By extending the space E_0 to E_k, a family of the exponential attractors of the initial-boundary value problem is obtained. In the second part, we consider the long-time behavior for a system of generalized Kirchhoff type with strong damping terms. Using the Hadamard graph transformation method, we obtain the existence of a family of the inertial manifolds while such equations satisfy the spectrum interval condition.

Keywords

A Family of the Exponential Attractors, Inertial Fractal Set, Squeezing Property, Spectral Gap Condition, A Family of the Inertial Manifolds

1. Introduction

Exponential attractor is a compact positive invariant set with finite fractal dimension and exponentially attracts every orbit, which is an important feature to describe the long-term behavior of nonlinear partial differential equations. In reference [1], since Foias and others put forward this concept in 1994, many mathematicians have made in-depth research on exponential attractors. Inertial manifold refers to the positive invariant Lipschitz manifold of finite dimension, which includes the global attractor attracting all solution orbits at exponential speed, and it is an important bridge between infinite dimensional dynamical system and finite dimensional dynamical system.

In reference [2], the author studied the exponential attractors of the following nonlinear wave equations by using operator decomposition and finite covering methods.

$$\begin{cases} u_{tt} + \alpha u_t - \Delta u + g(u) = f(x), (x,t) \in \Omega \times R^+, \\ u = 0, (x,t) \in \partial\Omega \times R^+, \\ u(x,0) = u_0(x), u_1(x,0) = u_1(x). \end{cases}$$

Contrary to the global attractor, the exponential attractor has a uniform exponential convergence rate on the invariant absorption set of its solution. Because of this, the exponential attractor has deeper and more practical properties, and under the perturbation and numerical approximation, the exponential attractor is more robust than the whole attractor.

In reference [3], Perikles G. Papadopoulos, Nikos M. Stavrakakis studied the global existence and blow-up of the following equations

$$u_{tt} - \phi(x)\left\|\nabla u(t)\right\|^2 \Delta u + \delta u_t = |u|^\alpha u, x \in R^N, t \geq 0$$

Initial condition $u(x,0) = u_0(x), u_t(x,0) = u_1(x)$.

Li *et al.* [4]. studied the global existence and blow-up of solutions for the following high-order Kirchhoff type equations with nonlinear dissipation terms

$$\begin{cases} u_{tt} + \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^q (-\Delta)^m u + u_t |u_t|^r = |u|^p u, x \in \Omega, t > 0; \\ u|_{\partial\Omega} = 0, \left.\frac{\partial^i u}{\partial v^i}\right|_{\partial\Omega} = 0, i = 1, \cdots, m-1, t > 0; \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega. \end{cases}$$

where $\Omega \subset R^N (N \geq 1)$ is a bounded open region with smooth boundary, υ is an outward normal vector, $m > 1$ is a positive integer and $p, q, r > 0$ is a normal number. In this paper, using the concavity method, it is obtained that the solution has global existence when $p \leq r$, but when $p > \max\{r, 2q\}$, for any initial value with negative initial energy, the solution explodes in a finite time with the norm in L^{p+2}. Salim [5] not only improves the results in reference [4] by modifying the proof method, but also proves that when the positive initial energy has an upper bound, the solution explodes in a finite time. Inspired by reference [4] [5], Ye *et al.* [6] studied the following hyperbolic equations of Kirchhoff type with damping term and source term:

$$\begin{cases} u_{tt} + \left\|A^{\frac{1}{2}} u\right\|^{2q} Au + a|u_t|^{q-2} u_t = b|u|^{r-2} u, x \in \Omega, t > 0; \\ u|_{\partial\Omega} = 0, \left.\frac{\partial^i u}{\partial v^i}\right|_{\partial\Omega} = 0, i = 1, \cdots, m-1, t > 0; \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega. \end{cases}$$

where $A = (-\Delta)^m$, $m > 1$ is a positive integer, $\Omega \subset R^N$ is a bounded region with smooth boundary, υ is an outward normal vector, and $a,b,p>0$ and $q,r>2$ are normal numbers. The author not only obtains the global existence of the solution by constructing a stable set in H_0^m, but also proves the estimation of energy attenuation by using Komornik lemma.

For more research on exponential attractors and inertial manifolds, we can read the literature [7]-[16].

Inspired by the above research, this paper will discuss a family of the existence of exponential attractors and inertial manifolds of a generalized Kirchhoff equation with damping term:

$$\begin{cases} u_{tt}+M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u_t+|u|^\rho(u_t+u)=f(x), \\ u(x,t)=0,\dfrac{\partial^i u}{\partial v^i}=0, i=1,2,\cdots,2m-1, x\in\partial\Omega, t>0, \\ u(x,0)=u_0(x), u_t(x,0)=u_t(x), x\in\Omega\subset R^n, \end{cases} \tag{1}$$

where $m\in N^+$, $\Omega\subset R^n(n\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $f(x)$ is an external force term, $M\left(\left\|\nabla^m u\right\|_p^p\right)$ is the stress term of Kirchhoff equation, $\beta>0$, $\beta(-\Delta)^{2m}u_t$ is a strong dissipative term, $|u|^\rho(u_t+u)$ is a nonlinear source term.

In this paper, our main difficulty is the handling of $M\left(\left\|\nabla^m u\right\|_p^p\right)$ and nonlinear terms $|u|^\rho(u_t+u)$. In order to overcome the difficulties, certain assumptions are needed to solve them. The algorithm of proof process has been used by predecessors. The previous algorithms are combined and extended to solve the difficulty of nonlinear term in the paper. This paper is organized as follows. Section 2 is some basic assumptions. Section 3 proves the existence a family of exponential attractors. Section 4 proves the existence of a family of the inertial manifolds by using the Hadamard graph transformation method.

2. Preliminaries

For brevity, we used the follow abbreviation:

$H=L^2(\Omega)$, $\|\cdot\|_{L^2(\Omega)}=\|\cdot\|$, $H_0^{2m}(\Omega)=H^{2m}(\Omega)\cap H_0^1(\Omega)$,

$H_0^{2m+k}(\Omega)=H^{2m+k}(\Omega)\cap H_0^1(\Omega)$, $E_k=H_0^{2m+k}(\Omega)\times H_0^k(\Omega),(k=0,1,2,\cdots,2m)$ and $C_i(i=0,1,2,\cdots)$ denotes positive constant, λ_1 is the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary condition on Ω.

The notation $(\cdot,\cdot),\|\cdot\|$ for the H inner product and norm,that is $(u,v)=\int_\Omega u(x)v(x)\mathrm{d}x$, $(u,u)=\|u\|^2$.

(H1) assume that Kirchhoff type stress term $M(s)\in C^2([0,+\infty],R)$ satisfies:

$$1<\mu_0\leq M(s)\leq\mu_1, \mu=\begin{cases}\mu_0, \dfrac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m}u\right\|^2\geq 0, \\ \mu_1, \dfrac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m}u\right\|^2<0.\end{cases}$$

where μ is a constant.

(H2) $\rho \le \frac{8m}{n}$.

3. Exponential Attractors

We denote the inner product and norm in E_k as following:

$$\forall U_i = (u_i, v_i) \in E_k, (i = 1,2),$$

we have

$$(U_1, U_2)_{E_k} = \left(\nabla^{2m+k} u_1, \nabla^{2m+k} u_2\right) + \left(\nabla^k v_1, \nabla^k v_2\right), \tag{1}$$

$$\|U\|_{E_k}^2 = (U,U)_{E_k} = \left\|\nabla^{2m+k} u\right\|^2 + \left\|\nabla^k v\right\|^2. \tag{2}$$

Setting $\forall U = (u,v)^{\mathrm{T}} \in E_k, v = u_t + \varepsilon u$, then Equation (1.1) can be converted into the following first-order evolution equation

$$U_t + H(U) = F(U), \tag{3}$$

where

$$H(U) = \begin{pmatrix} \varepsilon u - v \\ -\varepsilon v + \varepsilon^2 u + \beta(-\Delta)^{2m} v + (1-\beta\varepsilon)(-\Delta)^{2m} u \end{pmatrix} \tag{4}$$

$$F(U) = \begin{pmatrix} 0 \\ \left[1 - M\left(\left\|\nabla^m u\right\|_p^p\right)\right](-\Delta)^{2m} u - \left[|u|^\rho (u_t + u)\right] + f(x) \end{pmatrix} \tag{5}$$

In order to accomplish the proof, we need to construct a map. Let E_0, E_k are two Hilbert spaces with $E_k \to E_0$ is dense and continuous injection, and $E_k \to E_0$ is compact. Let $S(t)$ is a solution semigroup generated by Equation (3.3).

In the following definitions, $k = 1, 2, \cdots, 2m$.

Definition 3.1 [17] *A_k compact set $M_k \subset E_k$ is called an exponential attractor for $(S(t), B_k)$ if $A_k \subseteq M_k \subseteq B_k$ and*

1) $S(t) M_k \subseteq M_k, \forall t \ge 0$,

2) M_k has finite fractal dimension, $d_F(M_k) < +\infty$,

3) There exist universal constans C_0, C_1 such that

$$dist\left(S(t) B_k, M_k\right) \le C_0 \mathrm{e}^{-C_1 t}, \forall t > 0, \tag{6}$$

where $dist_{E_k}(A_k, B_k) = \sup_{x \in A_k} \inf_{y \in B_k} \|x - y\|_{E_k}$, B_k is a positively invariant set for $S(t)$ in E_k.

Definition 3.2 [17] *If for every $\delta \in \left(0, \frac{1}{8}\right)$, there exist a time $t^* > 0$, an integer $N_0 \ge 1$, and an orthogonal projection P_{N_0} of rank equal to N_0 such that for every U and V in B_k, either*

$$\left\|S(t^*)U - S(t^*)V\right\|_{E_k} \le \delta \|U - V\|_{E_k}, \tag{7}$$

or

$$\left\|Q_{N_0}\left(S(t^*)U - S(t^*)V\right)\right\|_{E_k} \le \left\|P_{N_0}\left(S(t^*)U - S(t^*)V\right)\right\|_{E_k}, \tag{8}$$

then we call $S(t)$ is squeezing in B_k, where $Q_{N_0} = I - P_{N_0}$.

Theorem 3.1 [1] *Assume that*

1) $S(t)$ possesses a family of (E_k, E_0)-compact attractors A_k,

2) $S(t)$ exists a positive invariant compact set $B_k \subset E_0$,

3) $S(t)$ is a Lipschitz continuous map with a Lipschitz continuous function $l(t)$ on B_k, such that $\|S(t)u - S(t)v\|_{E_k} \le l(t)\|u-v\|_{E_k}$, and satisfied the discrete squeezing property on B_k.

Then $S(t)$ has a family of (E_k, E_0)-compact exponential attractors M_k and

$$M_k = \bigcup_{0\le t\le t^*} S(t) M_k^*, \tag{9}$$

where

$$M_k^* = A_k \cup \left(\bigcup_{j=1}^{\infty} \bigcup_{i=1}^{\infty} S\left(t^*\right)^j \left(E^{(i)}\right) \right). \tag{10}$$

Moreover, the fractal dimension of M_k satisfies $d_F(M_k) \le 1 + cN_0$, where N_0 is the smallest N which make the discrete squeezing property established, $k = 1, 2, \cdots, 2m$.

Proposition 3.1 [1] *There exist* $t_0(D_k)$ *such that*

$$B_k = \overline{\bigcup_{0\le t\le t_0(D_k)} S(t) D_k}.$$

is the positive invariant set of $S(t)$ in E_0, and B_k attracts all bounded subsets of E_k, where B_k is a closed bounded absorbing set for $S(t)$ in E_k.

Proposition 3.2 [1] *Let* B_0, B_k *respectively are closed bounded absorbing set of Equation* (*3.3*) *in* E_0, E_k, *then* $S(t)$ *possesses a family of* (E_k, E_0)*-compact attractors* A_k.

Under of the appropriate hypothesized, the initial boundary value problem Equation (1.1) exists unique smooth. This solution possesses the following properties:

$$\|U\|_{E_0}^2 = \left\|\nabla^{2m} u\right\|^2 + \|v\|^2 \le C(R_1) \tag{11}$$

$$\|U\|_{E_k}^2 = \left\|\nabla^{2m+k} u\right\|^2 + \left\|\nabla^k v\right\|^2 \le C(R_2). \tag{12}$$

We denote the solution in Theorem 3.1 by $S(t)(U_0) = U$, the $S(t)$ is a continuous semigroup in E_0, There exist the balls:

$$D_0 = \left\{ U \in E_0 : \|U\|_{E_0}^2 \le C(R_1) \right\}, \tag{13}$$

$$D_k = \left\{ U \in E_k : \|U\|_{E_k}^2 \le C(R_2) \right\}, \tag{14}$$

respectively is a absorbing set of $S(t)$ in E_0 and E_k.

Lemma 3.1 *For* $\forall U = (u, v)^{\mathrm{T}} \in E_k$, *when we can obtain*

$$\left(H(U), U\right)_{E_k} \ge k_1 \|U\|_{E_k}^2 + k_2 \left\|\nabla^{2m+k} v\right\|^2. \tag{15}$$

Proof. By (3.1), (3.4) we get

$$
\begin{aligned}
&\left(H(U),U\right)_{E_k}\\
&=\left(\varepsilon\left(\nabla^{2m+k}u\right)-\nabla^{2m+k}v,\nabla^{2m+k}u\right)\\
&\quad+\left(-\varepsilon\nabla^k v+\varepsilon^2\nabla^k u+\beta(-\Delta)^{2m+\frac{k}{2}}v+(1-\beta\varepsilon)(-\Delta)^{2m+\frac{k}{2}}u,\nabla^k v\right)\\
&=\varepsilon\left\|\nabla^{2m+k}u\right\|^2-\varepsilon\left\|\nabla^k v\right\|^2+\varepsilon^2\left(\nabla^k u,\nabla^k v\right)+\beta\left\|\nabla^{2m+k}v\right\|^2\\
&\quad-\beta\varepsilon\left(\nabla^{2m+k}u,\nabla^{2m+k}v\right).
\end{aligned} \tag{16}
$$

By employing Hölder's inequality, Young's inequality and Poincaré's inequality, we process the terms in (3.16), we have

$$
\varepsilon^2\left(\nabla^k u,\nabla^k v\right)\geq-\frac{\varepsilon^2}{2}\left\|\nabla^k u\right\|^2-\frac{\varepsilon^2}{2}\left\|\nabla^k v\right\|^2\geq-\frac{\varepsilon^2}{2\lambda_1^{2m}}\left\|\nabla^{2m+k}u\right\|^2-\frac{\varepsilon^2}{2}\left\|\nabla^k v\right\|^2. \tag{17}
$$

$$
-\beta\varepsilon\left(\nabla^{2m+k}u,\nabla^{2m+k}v\right)\geq-\frac{\beta\varepsilon}{2}\left\|\nabla^{2m+k}u\right\|^2-\frac{\beta\varepsilon}{2}\left\|\nabla^{2m+k}v\right\|^2. \tag{18}
$$

By the value of ε ,and substituting (3.17)-(3.18), we have

$$
\begin{aligned}
\left(H(U),U\right)_{E_k}\geq&\left(\varepsilon-\frac{\varepsilon^2}{2\lambda_1^{2m}}-\frac{\beta\varepsilon}{2}\right)\left\|\nabla^{2m+k}u\right\|^2\\
&+\left(\frac{\beta\lambda_1^{2m}}{2}-\varepsilon-\frac{\varepsilon^2}{2}\right)\left\|\nabla^k v\right\|^2+\left(\frac{\beta}{2}-\frac{\beta\varepsilon}{2}\right)\left\|\nabla^{2m+k}v\right\|^2.
\end{aligned} \tag{19}
$$

because of $0<\varepsilon<\min\left\{2\lambda_1^{2m}-\beta\lambda_1^{2m},\sqrt{1+\beta\lambda_1^{2m}}-1,1\right\}$, so $\varepsilon-\frac{\varepsilon^2}{2\lambda_1^{2m}}-\frac{\beta\varepsilon}{2}\geq 0$, $\frac{\beta\lambda_1^{2m}}{2}-\varepsilon-\frac{\varepsilon^2}{2}\geq 0$, $\frac{\beta}{2}-\frac{\beta\varepsilon}{2}\geq 0$.

Let $k_1=\min\left\{\varepsilon-\frac{\varepsilon^2}{2\lambda_1^{2m}}-\frac{\beta\varepsilon}{2},\frac{\beta\lambda_1^{2m}}{2}-\varepsilon-\frac{\varepsilon^2}{2}\right\}$, $k_2=\frac{\beta}{2}-\frac{\beta\varepsilon}{2}$, we can get

$$
\left(H(U),U\right)_{E_k}\geq k_1\left\|U\right\|_{E_k}^2+k_2\left\|\nabla^{2m+k}v\right\|^2
$$

The proof is completed.

Let $S(t)U_0=U(t)=\left(u(t),v(t)\right)^{\mathrm{T}}$, where $v(t)=u_t(t)+\varepsilon u(t)$,

$S(t)V_0=V(t)=\left(\bar{u}(t),\bar{v}(t)\right)^{\mathrm{T}}$, where $\bar{v}(t)=\bar{u}_t(t)+\varepsilon\bar{u}(t)$,

Next set $\phi(t)=S(t)U_0-S(t)V_0=U(t)-V(t)=\left(w(t),z(t)\right)^{\mathrm{T}}$, where $z(t)=w_t(t)+\varepsilon w(t)$, then $\phi(t)$ satisfies:

$$
\phi_t(t)+H(U)-H(V)+F(V)-F(U)=0, \tag{20}
$$

$$
\phi_0(0)=U_0-V_0. \tag{21}
$$

In order to certify Equation (1.1) exists a family of exponential attractors,we first show the semigroup $S(t)$ of system (1.1) is Lipschitz continuous on B_k.

Lemma 3.2 (*Lipschitz property*) *For* $\forall U_0,V_0\in B_k$, *where* U_0,V_0 *is the initial values of problem* (1.1), *and* $t\geq 0$, *we have*

$$\left\|S(t)U_0 - S(t)V_0\right\|_{E_k}^2 \le \mathrm{e}^{rt}\left\|U_0 - V_0\right\|_{E_k}^2. \tag{22}$$

Proof. Taking the inner product of the Equation (3.20) with $\phi(t)$ in E_k, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\phi(t)\right\|_{E_k}^2 + \left(H(U) - H(V), \phi(t)\right)_{E_k} - \left((-\Delta)^{2m+\frac{k}{2}} w, \nabla^k z(t)\right)\\&+\left(M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m+\frac{k}{2}} u - M\left(\left\|\nabla^m \bar{u}\right\|_p^p\right)(-\Delta)^{2m+\frac{k}{2}} \bar{u}, \nabla^k z(t)\right)\\&+\left(\nabla^k\left(|u|^\rho (u_t + u) - |\bar{u}|^\rho(\bar{u}_t + \bar{u})\right), \nabla^k z(t)\right) = 0.\end{aligned} \tag{23}$$

Next, we deal with the following items one by one. Similar to Lemma 3.1, we easily obtain

$$\left(H(U) - H(V), \phi(t)\right)_{E_k} = \left(H(\phi(t)), \phi(t)\right)_{E_k} \ge k_1\left\|\phi(t)\right\|_{E_k}^2 + k_2\left\|\nabla^{2m+k} z(t)\right\|^2. \tag{24}$$

For convenience, let's call $s = \left\|\nabla^m u\right\|_p^p, \bar{s} = \left\|\nabla^m \bar{u}\right\|_p^p$, then by (H1) and using the mean value theorem, Young's inequality, we have

$$\begin{aligned}&\left(M\left(\left\|\nabla^m \bar{u}\right\|_p^p\right)(-\Delta)^{2m+\frac{k}{2}} \bar{u} - M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m+\frac{k}{2}} u, \nabla^k z(t)\right)\\&\le \left|\left(M(\bar{s})(-\Delta)^{2m+\frac{k}{2}} w, \nabla^k z(t)\right)\right| + \left|\left(M'(\zeta)(\bar{s} - s)(-\Delta)^{2m+\frac{k}{2}} u, \nabla^k z(t)\right)\right|\\&\le \frac{\mu_1}{2}\left\|\nabla^{2m+k} w\right\|^2 + \frac{\mu_1}{2}\left\|\nabla^{2m+k} z\right\|^2 + C_2\left\|\nabla^{2m+k} w\right\|\left\|\nabla^{2m+k} z\right\|\\&\le \frac{\mu_1 + C_2}{2}\left\|\nabla^{2m+k} w\right\|^2 + \frac{\mu_1 + C_2}{2}\left\|\nabla^{2m+k} z\right\|^2\end{aligned} \tag{25}$$

For the last term, we apply the mean value theorem, by (H2), we have

$$\begin{aligned}&\left(\nabla^k\left(|\bar{u}|^\rho(\bar{u}_t + \bar{u}) - |u|^\rho(u_t + u)\right), \nabla^k z(t)\right)\\&\le C_3\int_\Omega\left(|\bar{u}|^\rho + |u|^\rho\right)\mathrm{d}x\left|\nabla^k w_t\right|\left|\nabla^k z\right| + C_4\int_\Omega\left(|\bar{u}|^\rho + |u|^\rho\right)\mathrm{d}x\left|\nabla^k w\right|\left|\nabla^k z\right|\\&\le C_3\left(\left\||\bar{u}|^\rho\right\|_{L^\infty(\Omega)} + \left\||u|^\rho\right\|_{L^\infty(\Omega)}\right)\left(\frac{\left\|\nabla^k w_t\right\|^2}{2} + \frac{\left\|\nabla^k z\right\|^2}{2}\right)\\&\quad + C_4\left(\left\||\bar{u}|^\rho\right\|_{L^\infty(\Omega)} + \left\||u|^\rho\right\|_{L^\infty(\Omega)}\right)\left(\frac{\left\|\nabla^k w\right\|^2}{2} + \frac{\left\|\nabla^k z\right\|^2}{2}\right),\end{aligned}$$

By the interpolation inequality

$$\left\|\bar{u}\right\|_\infty^\rho \le C_5\left\|\nabla^{2m}\bar{u}\right\|^{\frac{\rho n}{4m}},$$

In the same way with

$$\left\|u\right\|_\infty^\rho \le C_6\left\|\nabla^{2m} u\right\|^{\frac{\rho n}{4m}},$$

where $\rho \le \frac{8m}{n}$.

Therefore

$$\begin{aligned}&\left(\nabla^k\left(|u|^\rho(u_t+u)-|\bar{u}|^\rho(\bar{u}_t+\bar{u})\right),\nabla^k z(t)\right)\\&\le C_7\left(\left\|\nabla^{2m}u\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\bar{u}\right\|^{\frac{\rho n}{4m}}\right)\left\|\nabla^k w_t\right\|^2\\&\quad+C_8\left(\left\|\nabla^{2m}u\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\bar{u}\right\|^{\frac{\rho n}{4m}}\right)\left\|\nabla^k z\right\|^2\\&\quad+C_9\left(\left\|\nabla^{2m}u\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\bar{u}\right\|^{\frac{\rho n}{4m}}\right)\left\|\nabla^k w\right\|^2\\&\le\left(\lambda_1^{2m}C_9\left(\left\|\nabla^{2m}u\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\bar{u}\right\|^{\frac{\rho n}{4m}}\right)\right)\left\|\nabla^{2m+k}w\right\|^2\\&\quad+C_8\left(\left\|\nabla^{2m}u\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\bar{u}\right\|^{\frac{\rho n}{4m}}\right)\left\|\nabla^k z\right\|^2.\end{aligned}\tag{26}$$

Integrating (3.24) - (3.26) into (3.23), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\phi(t)\right\|_{E_k}^2+k_1\left\|\phi(t)\right\|^2+\left(k_2-\frac{\mu_1+C_2}{2}-1\right)\left\|\nabla^{2m+k}z\right\|^2\\&\le\left(\frac{\mu_1+C_2}{2}+\lambda_1^{2m}C_9\left(\left\|\nabla^{2m}u\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\bar{u}\right\|^{\frac{\rho n}{4m}}\right)+1\right)\left\|\nabla^{2m+k}w\right\|^2\\&\quad+C_8\left(\left\|\nabla^{2m}u\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\bar{u}\right\|^{\frac{\rho n}{4m}}\right)\left\|\nabla^k z\right\|^2\\&\le C_{10}\left\|\phi(t)\right\|_{E_k}^2.\end{aligned}\tag{27}$$

where

$$C_{10}=\max\left\{\frac{\mu_1+C_2}{2}+\lambda_1^{2m}C_9\left(\left\|\nabla^{2m}u\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\bar{u}\right\|^{\frac{\rho n}{4m}}\right)-1,C_8\left(\left\|\nabla^{2m}u\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\bar{u}\right\|^{\frac{\rho n}{4m}}\right)\right\}.$$

By using Gronwall's inequality, we have

$$\left\|\phi(t)\right\|_{E_k}^2\le \mathrm{e}^{2C_{10}t}\left\|\phi(0)\right\|_{E_k}^2=\mathrm{e}^{rt}\left\|\phi(0)\right\|_{E_k}^2.\tag{28}$$

where $r=2C_{10}$, so we have

$$\left\|S(t)U_0-S(t)V_0\right\|_{E_k}^2\le \mathrm{e}^{rt}\left\|U_0-V_0\right\|_{E_k}^2.\tag{29}$$

The proved is completed.

Now,we introduce the operator $-\Delta$, Obviously, $-\Delta$ is an unbounded self-adjoin positive operator and $(-\Delta)^{-1}$ is compact. So, there is an orthonormal basis $\{w_i\}_{i=1}^{\infty}$ of H consisting of eigenvectors w_j of $-\Delta$ such that $(-\Delta)w_j=\lambda_j w_j$, $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_j\to+\infty$. $\forall N$ denote by $P=P_N$ the projector, P_N is an orthogonal projection, $Q=Q_N=I-P_N$.

As follows,we will need

$$\left\|(-\Delta)^{2m}u\right\|\ge\lambda_{N+1}^{2m}\left\|u\right\|,u\in Q_N\left(H^{2m}(\Omega)\cap H_0^1(\Omega)\right),$$

Lemma 3.3 *For* $\forall U_0,V_0\in B_k$, *where* U_0,V_0 *is the initial values of problem*

(1.1). *Let*

$$Q_{n_0}(t)=Q_{n_0}(U(t)-V(t))=Q_{n_0}(\phi(t))=\left(w_{n_0}(t),z_{n_0}(t)\right)^{\mathrm{T}}$$

then we have

$$\left\|\phi_{n_0}(t)\right\|_{E_k}^2\le\left(\mathrm{e}^{-2k_1t}+\frac{C_{19}\lambda_{N+1}^{2m}}{2k_1+r}\mathrm{e}^{rt}\right)\left\|\phi_{n_0}(0)\right\|_{E_k}^2. \tag{30}$$

Proof. Applying Q_{n_0} to (3.20), we have

$$\phi_{n_0t}(t)+Q_{n_0}(H(U)-H(V))+Q_{n_0}(F(V)-F(U))=0. \tag{31}$$

Taking the inner product of (3.31) with $Q_{n_0}(t)$ in E_k, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\phi_{n_0}(t)\right\|_{E_k}^2+k_1\left\|\phi_{n_0}(t)\right\|_{E_k}^2+k_2\left(\left\|\nabla^{2m+k}z_{n_0}(t)\right\|^2\right)\\&-\left((-\Delta)^{2m+\frac{k}{2}}w_{n_0}(t),\nabla^kz_{n_0}(t)\right)\\&+\left(Q_{n_0}\left(M\left(\left\|\nabla^mu\right\|_p^p\right)(-\Delta)^{2m+\frac{k}{2}}u-M\left(\left\|\nabla^m\bar{u}\right\|_p^p\right)(-\Delta)^{2m+\frac{k}{2}}\bar{u}\right),\nabla^kz_{n_0}(t)\right)\\&+\left(\nabla^k\left(Q_{n_0}\left(|u|^\rho(u_t+u)-|\bar{u}|^\rho(\bar{u}_t+\bar{u})\right)\right),\nabla^kz_{n_0}(t)\right)=0.\end{aligned} \tag{32}$$

Next, we deal with the following items one by one

$$\begin{aligned}&\left(Q_{n_0}\left(M\left(\left\|\nabla^m\bar{u}\right\|_p^p\right)(-\Delta)^{2m+\frac{k}{2}}\bar{u}-M\left(\left\|\nabla^mu\right\|_p^p\right)(-\Delta)^{2m+\frac{k}{2}}u\right),\nabla^kz_{n_0}(t)\right)\\&=\left(M\left(\bar{s}_{n_0}\right)(-\Delta)^{2m+\frac{k}{2}}\bar{u}_{n_0}-M\left(s_{n_0}\right)(-\Delta)^{2m+\frac{k}{2}}u_{n_0},\nabla^kz_{n_0}(t)\right)\\&\le\left|\left(M\left(\bar{s}_{n_0}\right)(-\Delta)^{2m+\frac{k}{2}}w_{n_0},\nabla^kz_{n_0}(t)\right)\right|\\&\quad+\left|\left(M'(\zeta)\left(\bar{s}_{n_0}-s_{n_0}\right)(-\Delta)^{2m+\frac{k}{2}}u_{n_0},\nabla^kz_{n_0}(t)\right)\right|\\&\le\frac{\mu_1}{2}\left\|\nabla^{2m+k}w_{n_0}\right\|^2+\frac{\mu_1}{2}\left\|\nabla^{2m+k}z_{n_0}\right\|^2+C_{11}\left\|\nabla^{2m+k}w_{n_0}\right\|\left\|\nabla^{2m+k}z_{n_0}\right\|\\&\le\left(\frac{\mu_1+C_{11}}{2}\right)\left\|\nabla^{2m+k}w_{n_0}\right\|^2+\left(\frac{\mu_1+C_{11}}{2}\right)\left\|\nabla^{2m+k}z_{n_0}\right\|^2.\end{aligned} \tag{33}$$

For the last term, we apply the mean value theorem, by (H2), we have

$$\begin{aligned}&\left(\nabla^k\left(Q_{n_0}\left(|\bar{u}|^\rho(\bar{u}_t+\bar{u})\right)-|u|^\rho(u_t+u)\right),\nabla^kz_{n_0}(t)\right)\\&=\left(\nabla^k\left(\left|\bar{u}_{n_0}\right|^\rho\left(\bar{u}_{n_0t}+\bar{u}_{n_0}\right)-\left|u_{n_0}\right|^\rho\left(u_{n_0t}+u_{n_0}\right)\right),\nabla^kz_{n_0}(t)\right)\\&\le C_{12}\left(\left\|\left|\bar{u}_{n_0}\right|^\rho\right\|_{L^\infty(\Omega)}+\left\|\left|u_{n_0}\right|^\rho\right\|_{L^\infty(\Omega)}\right)\left(\frac{\left\|\nabla^kw_{n_0t}\right\|^2}{2}+\frac{\left\|\nabla^kz_{n_0}\right\|^2}{2}\right)\\&\quad+C_{13}\left(\left\|\left|\bar{u}_{n_0}\right|^\rho\right\|_{L^\infty(\Omega)}+\left\|\left|u_{n_0}\right|^\rho\right\|_{L^\infty(\Omega)}\right)\left(\frac{\left\|\nabla^kw_{n_0}\right\|^2}{2}+\frac{\left\|\nabla^kz_{n_0}\right\|^2}{2}\right)\end{aligned}$$

By the interpolation inequality

$$\left\|\overline{u}_{n_0}\right\|_\infty^\rho \le C_{14}\left\|\nabla^{2m}\overline{u}_{n_0}\right\|^{\frac{\rho n}{4m}},$$

In the same way with

$$\left\|u_{n_0}\right\|_\infty^\rho \le C_{15}\left\|\nabla^{2m}u_{n_0}\right\|^{\frac{\rho n}{4m}},$$

where $\rho \le \frac{8m}{n}$.

Therefore

$$\begin{aligned}
&\left(\nabla^k\left(Q_{n_0}\left(\left|\overline{u}\right|^\rho\left(\overline{u}_t+\overline{u}\right)\right)-\left|u\right|^\rho\left(u_t+u\right)\right),\nabla^k z_{n_0}(t)\right)\\
&\le C_{16}\left(\left\|\nabla^{2m}u_{n_0}\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\overline{u}_{n_0}\right\|^{\frac{\rho n}{4m}}\right)\left\|\nabla^k w_{n_0t}\right\|^2\\
&\quad+C_{17}\left(\left\|\nabla^{2m}u_{n_0}\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\overline{u}_{n_0}\right\|^{\frac{\rho n}{4m}}\right)\left\|\nabla^k z_{n_0}\right\|^2\\
&\quad+C_{18}\left(\left\|\nabla^{2m}u_{n_0}\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\overline{u}_{n_0}\right\|^{\frac{\rho n}{4m}}\right)\left\|\nabla^k w_{n_0}\right\|^2\\
&\le\left(\lambda_{N+1}^{2m}C_{18}\left(\left\|\nabla^{2m}u_{n_0}\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\overline{u}_{n_0}\right\|^{\frac{\rho n}{4m}}\right)\right)\left\|\nabla^{2m+k}w_{n_0}\right\|^2\\
&\quad+C_{17}\left(\left\|\nabla^{2m}u_{n_0}\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\overline{u}_{n_0}\right\|^{\frac{\rho n}{4m}}\right)\left\|\nabla^k z_{n_0}\right\|^2
\end{aligned}\tag{34}$$

Integrating (3.33) - (3.34) into (3.32), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\phi_{n_0}(t)\right\|_{E_k}^2+k_1\left\|\phi_{n_0}(t)\right\|^2+\left(k_2-\frac{\mu_1+C_{11}}{2}-1\right)\left\|\nabla^{2m+k}z_{n_0}\right\|^2\\
&\le\left(\frac{\mu_1+C_{11}}{2}+\lambda_{N+1}^{2m}C_{18}\left(\left\|\nabla^{2m}u_{n_0}\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\overline{u}_{n_0}\right\|^{\frac{\rho n}{4m}}\right)+1\right)\left\|\nabla^{2m+k}w_{n_0}\right\|^2\\
&\quad+C_{17}\left(\left\|\nabla^{2m}u_{n_0}\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\overline{u}_{n_0}\right\|^{\frac{\rho n}{4m}}\right)\left\|\nabla^k z_{n_0}\right\|^2\\
&\le C_{19}\lambda_{N+1}^{2m}\left\|\phi_{n_0}(t)\right\|_{E_k}^2.
\end{aligned}\tag{35}$$

where

$$C_{19}=\max\left\{\left(\frac{\mu_1+C_{11}+2}{2\lambda_{N+1}^{2m}}+C_{18}\left(\left\|\nabla^{2m}u_{n_0}\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\overline{u}_{n_0}\right\|^{\frac{\rho n}{4m}}\right)\right),\frac{C_{17}\left(\left\|\nabla^{2m}u_{n_0}\right\|^{\frac{\rho n}{4m}}+\left\|\nabla^{2m}\overline{u}_{n_0}\right\|^{\frac{\rho n}{4m}}\right)}{\lambda_{N+1}^{2m}}\right\}.$$

Using Gronwall's inequality, we have

$$\left\|\phi_{n_0}(t)\right\|_{E_k}^2\le\left(\mathrm{e}^{-2k_1t}+\frac{C_{19}\lambda_{N+1}^{2m}}{2k_1+r}\mathrm{e}^{rt}\right)\left\|\phi_{n_0}(0)\right\|_{E_k}^2.\tag{36}$$

The proved is completed.

Lemma 3.4 (*squeezing property*) *For* $\forall U_0, V_0 \in B_k$, *if*

$$\left\| P_{n_0}\left(S\left(t^*\right)U_0 - S\left(t^*\right)V_0\right)\right\|_{E_k} \le \left\|\left(I - P_{n_0}\right)\left(S\left(t^*\right)U_0 - S\left(t^*\right)V_0\right)\right\|_{E_k}, \tag{37}$$

then we have

$$\left\| S\left(t^*\right)U_0 - S\left(t^*\right)V_0\right\|_{E_k} \le \frac{1}{8}\left\|U_0 - V_0\right\|_{E_k}. \tag{38}$$

Proof. If $\left\| P_{n_0}\left(S\left(t^*\right)U_0 - S\left(t^*\right)V_0\right)\right\|_{E_k} \le \left\|\left(I - P_{n_0}\right)\left(S\left(t^*\right)U_0 - S\left(t^*\right)V_0\right)\right\|_{E_k}$, then

$$\begin{aligned}
&\left\| S\left(t^*\right)U_0 - S\left(t^*\right)V_0\right\|_{E_k}^2 \\
&\le \left\| P_{n_0}\left(S\left(t^*\right)U_0 - S\left(t^*\right)V_0\right)\right\|_{E_k}^2 + \left\|\left(I - P_{n_0}\right)\left(S\left(t^*\right)U_0 - S\left(t^*\right)V_0\right)\right\|_{E_k}^2 \\
&\le 2\left\|\left(I - P_{n_0}\right)\left(S\left(t^*\right)U_0 - S\left(t^*\right)V_0\right)\right\|_{E_k}^2 \\
&\le 2\left(\mathrm{e}^{-2k_1 t^*} + \frac{C_{19}\lambda_{N+1}^{2m}}{2k_1 + r}\mathrm{e}^{rt^*}\right)\left\|U_0 - V_0\right\|_{E_k}^2.
\end{aligned} \tag{39}$$

Let t^* be large enough

$$\mathrm{e}^{-2k_1 t^*} \le \frac{1}{256}. \tag{40}$$

Also let n_0 be large enough

$$\frac{C_{19}\lambda_{N+1}^{2m}}{2k_1 + r}\mathrm{e}^{rt^*} \le \frac{1}{256}. \tag{41}$$

Subsituting (3.39) - (3.41) into (3.38), we have

$$\left\| S\left(t^*\right)U_0 - S\left(t^*\right)V_0\right\|_{E_k} \le \frac{1}{8}\left\|U_0 - V_0\right\|_{E_k}. \tag{42}$$

The proved is completed.

Theorem 3.2 *Under the above assumptions,* $U_0 \in E_k$, $k = 1, 2, \cdots, 2m$, $f \in H$ *.Then the initial boundary value problem* (1.1) *the solution semigroup has a family of* $\left(E_k, E_0\right)$ *-compact exponential attractors* M_k *on* B_k, $M_k = \bigcup_{0 \le t \le t^*} S(t)\left(A_k \cup \left(\bigcup_{j=1}^{\infty}\bigcup_{i=1}^{\infty} S\left(t^*\right)^j\left(E^{(i)}\right)\right)\right)$, *and the fractal dimension is satisfied* $d_F\left(M_k\right) \le 1 + cN_0$.

Proof. According to Theorem 3.1, Lemma 3.2, Theorem 3.2 is easily proven.

4. A Family of Inertial Manifolds

Next, we will prove the existence of a family of inertial manifolds when N is large enough by using graph norm transformation method.

Definition 4.1 [18] *Let* $S(t) = \left\{S(t)\right\}_{t \ge 0}$ *be the solution semigroup on Banach space* $E_k = H_0^{2m+k}(\Omega) \times H_0^k(\Omega)\left(k = 1, 2, \cdots, 2m\right)$, *and there is a subset* $\mu_k \subset E_k$:

1) μ_k is a finite-dimensional Lipschitz manifold;

2) μ_k is the positive invariant set, that is $S(t)\mu_k \subset \mu_k$, $\forall t > 0$;

3) μ_k attracts exponentially all orbits of solutions, that is, there are constants $\eta > 0$, $C > 0$, Such that

$$d\left(S(t)x, \mu_k\right) \le Ce^{-\eta t}, \forall t \ge 0, \forall x \in E_k;$$

It is said that μ_k is an inertial manifold about $S(t) = \{S(t)\}_{t\ge 0}$.

Definition 4.2 [18] *Let the operator* $\Lambda : E_k \to E_k$ *have several eigenvalues of positive real parts, and its eigenfunction* $\{w_j\}_{j\ge 1}$ *expands into the corresponding orthogonal space in* E_k, *and* $F \in C_b(E_k, E_k)$ *satisfies the Lipschitz condition*

$$\left\|F(U) - F(V)\right\|_{E_k} \le l_F \left\|U - V\right\|_{E_k}, U, V \in E_k. \tag{1}$$

If the point spectrum of the operator can be divided into two parts σ_1 and σ_2, where σ_1 is finite,

$$\Lambda_{k,1} = \sup\{\operatorname{Re}\lambda | \lambda \in \sigma_1\}, \Lambda_{k,2} = \sup\{\operatorname{Re}\lambda | \lambda \in \sigma_2\}, \tag{2}$$

$$E_{k_i} = span\{w_j | j \in \sigma_i\}, i = 1, 2. \tag{3}$$

Then

$$\Lambda_{k,2} - \Lambda_{k,1} > 4l_F, \tag{4}$$

$$E_k = E_{k_1} \oplus E_{k_2}, \tag{5}$$

hold with continuous orthogonal projection $P_{k,1} : E_k \to E_{k_1}$, $P_{k,2} : E_k \to E_{k_2}$, So it is said that the operator Λ satisfies the spectral interval condition, P is orthogonal projection.

Lemma 4.1 *Let the eigenvalues* $\mu_j^{\pm}, j \ge 1$ *is non-decreasing and for every* $n \in N$, *when* $N \ge n$, *such that* μ_N^- *and* μ_{N+1}^- *are consecutive adjacent values.*

Equation (1.1) are equivalent to the following first-order evolution equation:

$$U_t + \Lambda U = F(U), \tag{6}$$

with

$$U = (u, v)^{\mathrm{T}} \in E_k, v = u_t, \Lambda = \begin{pmatrix} 0 & -I \\ M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} & \beta(-\Delta)^{2m} \end{pmatrix}, \tag{7}$$

$$F(U) = \begin{pmatrix} 0 \\ f(x) - |u|^{\rho}(u_t + u) \end{pmatrix}. \tag{8}$$

we consider the graph norm on E_k, which induced by the scale product

$$(U, V)_{E_k} = \left(M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m+k} u, (-\Delta)^{2m+k} \overline{y}\right) + \left(\nabla^k \overline{g}, \nabla^k v\right), \tag{9}$$

where $U = (u, v)^{\mathrm{T}}, V = (y, g)^{\mathrm{T}} \in E_k$; $\overline{y}, \overline{g}$ represent the conjugation of y, g respectively; $u, v, y, g \in H^{2m+k}(\Omega)$. Obviously, the operator Λ defined in (4.2) is monotone. Indeed, for $U \in E_k$,

$$\begin{aligned}(\Lambda U,U)_{E_k} &= \left(\left(-v, M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}v\right),(u,v)\right)\\ &= \left(-M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m+k}v,(-\Delta)^{2m+k}\bar{u}\right)\\ &\quad+\left(\nabla^k\bar{v},\nabla^k\left(M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}v\right)\right)\\ &= \beta\left\|\nabla^{2m+k}v\right\|^2\geq 0.\end{aligned} \tag{10}$$

Therefore, $(\Lambda U,U)_{E_k}$ is a non-negative real number.

In order to determine the characteristic value of Λ, we consider the following characteristic equation

$$\Lambda U=\lambda U, \forall U=(u,v)^{\mathrm{T}}\in E_k, \tag{11}$$

that is

$$\begin{cases}-v=\lambda u,\\ M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}v=\lambda v.\end{cases} \tag{12}$$

Substituting the first Equation of (4.12) into the second equation can be obtained

$$\lambda^2 u+M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m}u-\beta\lambda(-\Delta)^{2m}u=0. \tag{13}$$

Taking the inner product of $(-\Delta)^k u$, on both sides of the Equations of (4.13) respectively, we acquire

$$\lambda^2\left\|\nabla^k u\right\|^2+M\left(\|\nabla^m u\|_p^p\right)\left\|\nabla^{2m+k}u\right\|^2-\beta\lambda\left\|\nabla^{2m+k}u\right\|^2=0. \tag{14}$$

Regarding (4.14) as a quadratic equation of one variable with respect to λ, for $j\in N^+$, and let $s=\|\nabla^m u\|_p^p$, $M=M(s)$, the corresponding eigenvalues of Equation (4.11) are as follows:

$$\lambda_j^{\pm}=\frac{\beta\xi_j\pm\sqrt{\beta^2\xi_j^2-4M\left(\|\nabla^m u\|_p^p\right)\xi_j}}{2}, \tag{15}$$

where $j\geq 1$, ξ_j is the eigenvalue of $(-\Delta)^{2m}$ in $H_0^{2m}(\Omega)$, then $\xi_j=\lambda_j j^{\frac{2m}{n}}$. If $\xi_j\geq\frac{4M\left(\|\nabla^m u\|_p^p\right)}{\beta^2}$, then $\xi_j\geq\frac{4\mu_1}{\beta^2}$, that is all the eigenvalues of Λ are positive real numbers, and the corresponding eigenvectors are in the form of $U_j^{\pm}=\left(u_j,-\lambda_j^{\pm}u_j\right)$. For convenience, we note that for any $j\geq 1$,

$$\left\|\nabla^{2m+k}u_j\right\|=\sqrt{\xi_j},\left\|\nabla^k u_j\right\|^2=1,\left\|\nabla^{-(2m+k)}u_j\right\|=\frac{1}{\sqrt{\xi_j}},k=1,2,\cdots,2m. \tag{16}$$

Theorem 4.1 *Assue* $\xi_j\geq\frac{4M(s)}{\beta^2}$, $N_1\in N^+$ *is large enough, when* $N>N_1$, *the following inequality holds*

$$\left(\xi_{N+1}-\xi_N\right)\left(\beta-\sqrt{\beta^2\xi_j-4M(s)}-1\right)$$
$$\geq 8\left(\left(C_{20}\lambda_1^{-m}\left(C_{21}\left\|\nabla^{2m}\hat{u}\right\|^{\frac{\rho n}{4m}}+C_{22}\left\|\nabla^{2m}u\right\|^{\frac{\rho n}{4m}}\right)+1\right)\right). \tag{17}$$

Then the operator Λ satisfies the spectral gap condition $\Lambda_{k,2}-\Lambda_{k,1}>4l_F$.

Proof: It is known that all the eigenvalues of Λ are positive real numbers, $\beta\geq 2\sqrt{\frac{M(s)}{\xi_j}}$, and the sequence $\left\{\lambda_j^-\right\}_{j\geq 1}$ and $\left\{\lambda_j^+\right\}_{j\geq 1}$ are monotonically increasing.

The following four steps to prove Theorem 4.1.

step 1: Because $\lambda_j^{\pm}$ is a non-decreasing sequence. According to Lemma 4.1, given N such that λ_N^- and λ_{N+1}^- are consecutive adjacent eigenvalues, the eigenvalues of the operator Λ are decomposed into δ_1 and δ_2, where δ_1 is the finite parts, which are expressed as follows.

$$\delta_1=\left\{\lambda_s^-,\lambda_j^+\middle|\max\left\{\lambda_s^-,\lambda_j^+\leq\lambda_N^-\right\}\right\}, \tag{18}$$

$$\delta_2=\left\{\lambda_s^+,\lambda_j^{\pm}\middle|\lambda_s^-\leq\lambda_N^-\leq\min\left\{\lambda_s^+,\lambda_j^{\pm}\right\}\right\}. \tag{19}$$

step 2: Consider the corresponding decomposition of E_k.

$$E_{k_1}=span\left\{U_s^-,U_j^{\pm}\middle|\lambda_s^-,\lambda_j^+\in\delta_1\right\}, \tag{20}$$

$$E_{k_2}=span\left\{U_s^+,U_j^+\middle|\lambda_s^-,\lambda_j^{\pm}\in\delta_2\right\}. \tag{21}$$

The purpose is to make these two orthogonal subspaces of E_k and satisfy the spectral gap Equation (4.4) is true when $\Lambda_{k,1}=\lambda_N^-$, $\Lambda_{k,2}=\lambda_{N+1}^-$. Further decomposition E_{k_2}, then $E_{k_2}=E_{k_C}\oplus E_{k_R}$,

$$E_{k_C}=span\left\{U_s^-\middle|\lambda_s^-\leq\lambda_N^-\leq\lambda_s^+\right\}, \tag{22}$$

$$E_{k_R}=span\left\{U_R^+\middle|\lambda_N^-\leq\lambda_j^{\pm}\right\}, \tag{23}$$

and set $E_{k_N}=E_{k_C}\oplus E_{k_1}$. Note that E_{k_1} and E_{k_C} are finite dimensional, that $\lambda_N^-\in E_{k_1}$, $\lambda_{N+1}^-\in E_{k_R}$, and that the reason why E_{k_1} is not orthogonal to E_{k_2} is that, while it is orthogonal to E_{k_R} is not orthogonal to E_{k_C}

Now we introduce two functions $\Phi:E_{k_N}\to R$, $\Psi:E_{k_R}\to R$, defined by

$$\begin{aligned}\Phi(U,V)=&\,2\beta^2\left(\nabla^{2m+k}u,\nabla^{2m+k}\overline{y}\right)+2\beta\left(\nabla^{-(2m+k)}\overline{g},\nabla^{2m+k}u\right)\\&+2\beta\left(\nabla^{-(2m+k)}\overline{v},\nabla^{2m+k}y\right)+4\left(\nabla^{-(2m+k)}\overline{v},\nabla^{-(2m+k)}g\right)\\&-4M\left(\left\|\nabla^m u\right\|_p^p\right)\left(\nabla^k u,\nabla^k\overline{y}\right),\end{aligned} \tag{24}$$

$$\begin{aligned}\Psi(U,V)=&\,2\beta^2\left(\nabla^{2m+k}u,\nabla^{2m+k}\overline{y}\right)+\beta\left(\nabla^{-(2m+k)}\overline{g},\nabla^{2m+k}u\right)\\&+\beta\left(\nabla^{-(2m+k)}\overline{v},\nabla^{2m+k}y\right)+4\left(\nabla^{-(2m+k)}\overline{g},\nabla^{-(2m+k)}v\right),\end{aligned} \tag{25}$$

with $U=(u,v),V=(y,g)$, $\bar{y},\bar{g}$ represents the conjugate of y and g respectively.

For $U=(u,v)\in E_{k_N}$, then

$$\begin{aligned}\Phi(U,U)&=2\beta^2\left(\nabla^{2m+k}u,\nabla^{2m+k}\bar{u}\right)+2\beta\left(\nabla^{-(2m+k)}\bar{v},\nabla^{2m+k}u\right)\\&\quad+2\beta\left(\nabla^{-(2m+k)}\bar{v},\nabla^{2m+k}u\right)+4\left(\nabla^{-(2m+k)}\bar{v},\nabla^{-(2m+k)}v\right)\\&\quad-4M\left(\left\|\nabla^m u\right\|_p^p\right)\left(\nabla^k u,\nabla^k\bar{u}\right)\\&\ge 2\beta^2\left\|\nabla^{2m+k}u\right\|^2-2\left\|\nabla^{-(2m+k)}v\right\|^2-\frac{\beta^2}{2}\left\|\nabla^{2m+k}u\right\|^2+4\left\|\nabla^{-(2m+k)}v\right\|^2\\&\quad-4M\left(\left\|\nabla^m u\right\|_p^p\right)\left\|\nabla^k u\right\|^2-2\left\|\nabla^{-(2m+k)}v\right\|^2-\frac{\beta^2}{2}\left\|\nabla^{2m+k}u\right\|^2\\&=\beta^2\left\|\nabla^{2m+k}u\right\|^2-4M\left(\left\|\nabla^m u\right\|_p^p\right)\left\|\nabla^k u\right\|^2\\&\ge\left(\beta^2\xi_1-4M\left(\left\|\nabla^m u\right\|_p^p\right)\right)\left\|\nabla^k u\right\|^2,\end{aligned}\tag{26}$$

For any k, there is $\beta^2\xi_k\ge 4M(\xi_k)$, and according to the initial hypothesis $\mu_0\le M(s)\le\mu_1\le\frac{\beta^2\xi_k}{4}$, that is $\Phi(U,U)\ge 0$, Φ is positive definite. Similarly, for $U=(u,v)\in E_{k_R}$, then

$$\begin{aligned}\Psi(U,U)&=2\beta^2\left(\nabla^{2m+k}u,\nabla^{2m+k}\bar{u}\right)+\beta\left(\nabla^{-(2m+k)}\bar{v},\nabla^{2m+k}u\right)\\&\quad+\beta\left(\nabla^{-(2m+k)}v,\nabla^{2m+k}u\right)+4\left(\nabla^{-(2m+k)}\bar{v},\nabla^{-(2m+k)}v\right)\\&\ge 2\beta^2\left\|\nabla^{2m+k}u\right\|^2+4\left\|\nabla^{-(2m+k)}v\right\|^2-\beta^2\left\|\nabla^{2m+k}u\right\|^2-4\left\|\nabla^{-(2m+k)}v\right\|^2\\&\ge\beta^2\xi_1\left\|\nabla^k u\right\|^2.\end{aligned}\tag{27}$$

that is $\Psi(U,U)\ge 0$, Ψ is positive definite.

Thus Φ and Ψ define a scalar product, respectively on E_{k_N} and E_{k_C}, and we can define an equivalent scalar product in E_k, by

$$\langle\langle U,V\rangle\rangle_{E_k}=\Phi(P_N U,P_N V)+\Psi(P_R U,P_R V).\tag{28}$$

where P_N and P_R are projections of E_k to E_{k_N} and E_{k_R} respectively, for brief, (4.28) can be abbreviated as the following

$$\langle\langle U,V\rangle\rangle_{E_k}=\Phi(U,V)+\Psi(U,V).$$

We proved then to show that the subspaces E_{k_1} and E_{k_2} defined in (4.20), (4.21) are orthogonal with respect to the scalar product (4.28). In fact, it is sufficient to show that E_{k_N} is orthogonal to E_{k_C}, in turn, this reduces to showing that

$$\left\langle\left\langle U_s^+,U_s^-\right\rangle\right\rangle_{E_k}=\Phi\left(U_s^+,U_s^-\right)=0\left(\forall U_s^+\in E_{k_C},U_s^-\in E_{k_N}\right).\tag{29}$$

Recalling (4.26) and (4.27), $\forall U_s^+\in E_{k_C},U_s^-\in E_{k_N}$

$$\begin{aligned}\Phi\left(U_s^+,U_s^-\right)&=2\beta^2\left(\nabla^{2m+k}u_s,\nabla^{2m+k}\overline{u}_s\right)+2\beta\left(-\lambda_s^+\nabla^{-(2m+k)}\overline{u}_s,\nabla^{2m+k}u_s\right)\\&\quad+2\beta\left(-\lambda_s^-\nabla^{-(2m+k)}\overline{u}_s,\nabla^{-(2m+k)}u_s\right)\\&\quad+4\left(-\lambda_s^+\nabla^{-(2m+k)}\overline{u}_s,-\lambda_s^-\nabla^{-(2m+k)}u_s\right)-4M(s)\left(\nabla^k u_s,\nabla^k\overline{u}_s\right)\\&=2\beta^2\left\|\nabla^{2m+k}u_s\right\|^2-2\beta\left(\lambda_s^-+\lambda_s^+\right)\left\|u_s\right\|^2\\&\quad+4\lambda_s^+\lambda_s^-\left\|\nabla^{-(2m+k)}u_s\right\|^2-4M(s)\left\|\nabla^k u_s\right\|^2\\&=2\beta^2\xi_s-2\beta\left(\lambda_s^-+\lambda_s^+\right)+4\lambda_s^+\lambda_s^-\frac{1}{\xi_s}-4M\left(\xi_s\right).\end{aligned}\tag{30}$$

according to (4.15)

$$\lambda_s^-+\lambda_s^+=\beta\xi_s,\lambda_s^+\lambda_s^-=M\left(\xi_s\right)\xi_s,$$

thus, (4.30) is equivalent to

$$\Phi\left(U_s^+,U_s^-\right)=2\beta^2\xi_s-2\beta\left(\lambda_s^-+\lambda_s^+\right)+4\lambda_s^+\lambda_s^-\frac{1}{\xi_s}-4M\left(\xi_s\right)=0.$$

step 3: Further, we estimate the Lipschitz constant l_F of

$$F(U)=\left(0,f(x)-|u|^\rho\left(u_t+u\right)\right)^{\mathrm{T}},$$

$$\forall U=(u,v)^{\mathrm{T}}\in E_k,U_1=\left(u_1,v_1\right)^{\mathrm{T}}=P_{k,1}U,U_2=\left(u_2,v_2\right)^{\mathrm{T}}=P_{k,2}U,$$

then

$$P_{k,1}u=u_1,P_{k,2}u=u_2.$$

Give $U=(u,v),V=(\hat{u},\hat{v})\in E_k$, we get

$$\begin{aligned}\left\|F(U)-F(V)\right\|_{E_k}&=\left\|\left(0,|\hat{u}|^\rho\left(\hat{u}_t+\hat{u}\right)-|u|^\rho\left(u_t+u\right)\right)\right\|\\&=\left\|\nabla^k\left(|\hat{u}|^\rho\hat{u}_t-|u|^\rho u_t\right)\right\|+\left\|\nabla^k\left(|\hat{u}|^\rho\hat{u}-|u|^\rho u\right)\right\|\\&\le C_{20}\left(\|\hat{u}\|_\infty^\rho+\|u\|_\infty^\rho\right)\left(\left\|\nabla^k\left(\hat{u}_t-u_t\right)\right\|+\left\|\nabla^k\left(\hat{u}-u\right)\right\|\right)\end{aligned}$$

By the interpolation inequality

$$\|\hat{u}\|_\infty^\rho\le C_{21}\left\|\nabla^{2m}\hat{u}\right\|^{\frac{\rho n}{4m}}$$

$$\|u\|_\infty^\rho\le C_{22}\left\|\nabla^{2m}u\right\|^{\frac{\rho n}{4m}}$$

where $\rho\le\dfrac{8m}{n}$.

Therefore

$$\begin{aligned}&\left\|F(U)-F(V)\right\|_{E_k}\\&\le\left(C_{20}\lambda_1^{-m}\left(C_{21}\left\|\nabla^{2m}\hat{u}\right\|^{\frac{\rho n}{4m}}+C_{22}\left\|\nabla^{2m}u\right\|^{\frac{\rho n}{4m}}\right)+1\right)\left(\left\|\nabla^{2m+k}\left(\hat{u}_t-u_t\right)\right\|+\left\|\nabla^k\left(\hat{u}-u\right)\right\|\right)\\&\le\left(\left(C_{20}\lambda_1^{-m}\left(C_{21}\left\|\nabla^{2m}\hat{u}\right\|^{\frac{\rho n}{4m}}+C_{22}\left\|\nabla^{2m}u\right\|^{\frac{\rho n}{4m}}\right)+1\right)\right)\left\|U-V\right\|_{E_k}\end{aligned}$$

thus

$$l_F \leq \left(\left(C_{20} \lambda_1^{-m} \left(C_{21} \left\| \nabla^{2m} \hat{u} \right\|^{\frac{\rho n}{4m}} + C_{22} \left\| \nabla^{2m} u \right\|^{\frac{\rho n}{4m}} \right) + 1 \right) \right) \tag{31}$$

step 4: Now we need to verify that the spectral interval condition $\Lambda_2 - \Lambda_1 > 4l_F$ is established. $\Lambda_1 = \lambda_N^-$ and $\Lambda_1 = \lambda_{N+1}^-$, we can get

$$\Lambda_2 - \Lambda_1 = \lambda_{N+1}^- - \lambda_N^- = \frac{\beta}{2}\left(\xi_{N+1} - \xi_N\right) + \frac{1}{2}\left(\sqrt{R(N)} - \sqrt{R(N+1)}\right), \tag{32}$$

with $R(N) = \beta^2 \xi_N^2 - 4M(s)\xi_N$.

and

$$\lim_{N \to +\infty} \left(\sqrt{R(N)} - \sqrt{R(N+1)} + \sqrt{\beta^2 \xi_1 - 4M(s)} \right) \left(\xi_{N+1} - \xi_N \right) = 0. \tag{33}$$

For formula (4.32). There, $\exists N_1 > 0$, such that for $\forall N > N_1$,

$$R_1(N) = \frac{\beta^2 \xi_N - 4M(s)}{\xi_N \left(\beta^2 \xi_1 - 4M(s) \right)}$$

we can get

$$\begin{aligned} &\sqrt{R(N)} - \sqrt{R(N+1)} + \sqrt{\beta^2 \xi_1 - 4M(s)} \left(\xi_{N+1} - \xi_N \right) \\ &= \sqrt{\beta^2 \xi_1 - 4M(s)} \left(\xi_{N+1} \left(1 - \sqrt{R_1(N+1)} \right) - \xi_N \left(1 - \sqrt{R_1(N)} \right) \right), \end{aligned} \tag{34}$$

$$\lim_{N \to +\infty} \xi_N \left(1 - \sqrt{R_1(N)} \right) = 0.$$

From the condition, it can be determined that $N_1 > 0$ such that for all $N \geq N_1$, and with (4.32)

$$\begin{aligned} \Lambda_2 - \Lambda_1 = \lambda_{N+1}^- - \lambda_N^- &\geq \frac{\xi_{N+1} - \xi_N}{2} \left(\beta - \sqrt{\beta^2 \xi_1 - 4M(s)} \right) - 1 \\ &\geq 4 \left(\left(C_{20} \lambda_1^{-m} \left(C_{21} \left\| \nabla^{2m} \hat{u} \right\|^{\frac{\rho n}{4m}} + C_{22} \left\| \nabla^{2m} u \right\|^{\frac{\rho n}{4m}} \right) + 1 \right) \right) \geq 4l_F \end{aligned} \tag{35}$$

under the latter assumption, Theorem 4.1 is proved completely.

Theorem 4.2 *In the conclusions of Theorem* 4.1, *initial boundary value problems admits an inertial manifold* μ_k *in* E_k *of the form*

$$\mu_k = graph(\Gamma) = \left\{ \zeta_k + \Gamma(\zeta_k) : \zeta_k \in E_{k_1} \right\}, \tag{36}$$

where $\Gamma : E_{k_1} \to E_{k_2}$ is Lipschitz continuous with the Lipschitz constant l_F ,and $graph(\Gamma)$ represents the diagram of Γ.

Proof: According to Theorem 4.1, Lemma 4.1 and Definition 4.1 is easily proven.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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