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On the Field Equations of General Relativity

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Abstract

In this work, we examine the geometric character of the field equations of general relativity and propose to formulate relativistic field equations in terms of the Riemann curvature tensor. The resulted relativistic field equations are also integrated into the general framework that we have presented in our previous works that all known classical fields can be expressed in the same dynamical form. We also discuss a possibility to reformulate the field equations of general relativity so that the Ricci curvature tensor and the energy-momentum tensor can appear symmetrically in the field equations without violating the conservation law stated by the covariant derivative.

Keywords

General Relativity, Classical Field Equations, Riemann Curvature Tensor

1. Introduction

Perhaps, one of the most insightful features that emerges from Einstein field equations of general relativity, $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = kT_{\alpha\beta}$, is that matter and the spacetime continuum should be presented with an intrinsic geometric structure of a differentiable manifold [1] [2]. However, apparently, the Einstein general relativistic field equations do not fully describe the assumed mathematical structure of matter and the spacetime continuum, because the mathematical object that involves in the equations is the Ricci curvature tensor $R_{\alpha\beta}$ instead of the Riemann curvature tensor $R^{\alpha}_{\beta\mu\nu}$. If matter and the spacetime continuum are endowed with intrinsic geometric structures of differentiable manifolds, in which we may assume geometric objects manifest as physical entities, then it is obvious that physical laws should also be formulated with the application of mathematical objects that can carry the most intrinsic property of a differentiable manifold, and categorically such mathematical object is the Riemann curvature tensor $R^{\alpha}_{\beta\mu\nu}$. Despite the fact that for differentiable manifolds with dimension $n = 3$, the Ricci curvature tensor can play the role of such complete mathematical object, because in this case the Riemann curvature tensor can be composed entirely in terms of the Ricci curvature tensor. However, differentiable manifolds with $n \geq 4$ are not. This can be seen by using the decomposition of the Riemann curvature tensor into the Ricci tensor and the Weyl tensor $C_{\alpha\beta\mu\nu}$ [3] [4]

$$R_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu} + \frac{1}{n-2}\left(g_{\alpha\mu}R_{\nu\beta} + g_{\beta\nu}R_{\mu\alpha} - g_{\alpha\nu}R_{\mu\beta} - g_{\beta\mu}R_{\nu\alpha}\right) - \frac{1}{(n-1)(n-2)}\left(g_{\alpha\mu}R_{\nu\beta} - g_{\alpha\nu}R_{\mu\beta}\right)R \tag{1}$$

and for three-dimensional differentiable manifolds $C_{\alpha\beta\mu\nu} \equiv 0$. Therefore, in this work, we will discuss a possibility to formulate relativistic field equations using the Riemann curvature tensor instead of the Ricci curvature tensor. For differentiable manifolds with $n \geq 4$, we would need to establish physical laws directly in terms of the Riemann curvature tensor in order to completely describe the intrinsic properties of physical objects that are represented by the mathematical objects characterized by the mathematical structures of differentiable manifolds. Nevertheless, as shown in Section 2, we still need Einstein field equations of general relativity, formulated in terms of the Ricci curvature tensor, to establish the required relationship between the Riemann curvature tensor and the energy-momentum tensor. On the other hand, in Section 3, we will discuss a possibility to extend the concept of the energy-momentum tensor so that this essential physical entity should also be required to possess the mathematical structure of the Riemann curvature tensor.

The present formulation of relativistic field equations is also constructed in line with the framework of our previous works that show that classical physics can be formulated from system of field equations written in the general form [5] [6] [7] [8]

$$\nabla_{\mu} M = J \tag{2}$$

where M is a mathematical object, J a physical entity identified from a geometrical object, and ∇_{β} a covariant derivative. We have shown that classical physics can be formulated using the general form given in Equation (2). For Newton classical mechanics, we set $M = E$ with $E = (m/2)\sum_{\mu=1}^{3}\left(\mathrm{d}x^{\mu}/\mathrm{d}t\right)^2 + V$ and $J = 0$. For Maxwell electromagnetic field, we set $M = F^{\alpha\beta}$, where the electromagnetic tensor $F^{\alpha\beta}$ expressed in terms of the four-vector potential $A^{\mu} \equiv (V, \boldsymbol{A})$ as $F^{\alpha\beta} = \partial A^{\beta}/\partial x^{\alpha} - \partial A^{\alpha}/\partial x^{\beta}$ with the four-current $j^{\beta} \equiv (\rho_e, \boldsymbol{j}_e)$. For the gravitational field, we set $M = R^{\alpha\beta}$, and in this case, the quantity J in Equation (2) is defined purely in terms of geometrical objects as $j^{\alpha} = \frac{1}{2} g^{\alpha\beta}\nabla_{\beta}R$. Furthermore, the energy-momentum tensor $T_{\alpha\beta}$ for the gravitational field can be established in terms of the Ricci curvature tensor $R_{\alpha\beta}$ and the metric tensor $g_{\alpha\beta}$ as

$$kT_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \tag{3}$$

The system of equations given in Equation (3) is the Einstein field equations of general relativity. It should be mentioned here that using the Einstein field equations, the purely geometric current $j^{\alpha} = \frac{1}{2} g^{\alpha\beta} \nabla_{\beta} R$ can be given a physical meaning expressed in terms of the energy-momentum tensor. The Equation (3) can be rewritten in the form

$$R_{\alpha\beta} = k\left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T\right) \tag{4}$$

where $T = g^{\alpha\beta} T_{\alpha\beta}$. Then using $\nabla_{\mu} g^{\alpha\beta} = 0$ and $\nabla_{\beta} T^{\alpha\beta} = 0$, we obtain the equation

$$\nabla_{\beta} R^{\alpha\beta} = j^{\alpha} \tag{5}$$

where $j^{\alpha} = -(1/2k) g^{\alpha\beta} \nabla_{\beta} T$.

In the next section, we will show that relativistic field equations formulated in terms of the Riemann curvature tensor also take the form presented by the general equation given in Equation (2).

2. Formulation of Relativistic Field Equations Using Riemann Curvature Tensor

In this section, we discuss the possibility to formulate relativistic field equations that can be used to describe physical fields using the Riemann curvature tensor $R^{\alpha}_{\beta\mu\nu}$. Within the present formulation, the constructed field equations, as shown below, could be assumed to be a description of the gravitational field which manifests as a curved spacetime with a source represented by an energy-momentum tensor $T_{\alpha\beta}$ and the Ricci curvature tensor $R_{\alpha\beta}$ can be established in terms of the tensor $T_{\alpha\beta}$ according to the relation given in Equation (4).

In differential geometry, the Riemann curvature tensor $R^{\alpha}_{\beta\mu\nu}$ can be written in terms of the affine connection $\Gamma^{\alpha}_{\beta\lambda}$

$$R^{\alpha}_{\beta\mu\nu} = \frac{\partial \Gamma^{\alpha}_{\nu\beta}}{\partial x^{\mu}} - \frac{\partial \Gamma^{\alpha}_{\mu\beta}}{\partial x^{\nu}} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\nu\beta} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\beta} \tag{6}$$

By contraction using the metric tensor $g_{\alpha\beta}$ then we obtain the covariant Riemann curvature tensor $R_{\alpha\beta\mu\nu}$, the Ricci curvature tensor $R_{\alpha\beta}$, and the Ricci scalar curvature R

$$R_{\alpha\beta\mu\nu} = g_{\alpha\gamma} R^{\gamma}_{\beta\mu\nu} \tag{7}$$

$$R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta} \tag{8}$$

$$R = g^{\alpha\beta} R_{\alpha\beta} \tag{9}$$

And, importantly, it is shown that the covariant Riemann curvature tensor $R_{\alpha\beta\mu\nu}$ satisfies the Bianchi identities [9]

$$\nabla_{\lambda} R_{\alpha\beta\mu\nu} + \nabla_{\nu} R_{\alpha\beta\lambda\mu} + \nabla_{\mu} R_{\alpha\beta\nu\lambda} = 0 \tag{10}$$

Using the identities $\nabla_{\mu} g^{\alpha\beta} = 0$, the Bianchi identities given in Equation (10) can be contracted by $g^{\alpha\mu}$ to give the following relation between the Riemann curvature tensor and the Ricci curvature tensor

$$\nabla_\lambda R_{\beta\nu} - \nabla_\nu R_{\beta\lambda} + \nabla_\mu R^\mu_{\beta\nu\lambda} = 0 \quad (11)$$

By rearranging Equation (11) then we obtain the required relativistic field equations in terms of the Riemann curvature tensor that take the general form given in Equation (2) and are given as follows

$$\nabla_\mu R^\mu_{\beta\nu\lambda} = J_{\beta\nu\lambda} \quad (12)$$

where the quantity $J_{\beta\nu\lambda}$ is defined as

$$J_{\beta\nu\lambda} = \nabla_\nu R_{\beta\lambda} - \nabla_\lambda R_{\beta\nu} \quad (13)$$

The current $J_{\beta\nu\lambda}$ defined in terms of the covariant derivatives of the Ricci curvature tensor is still purely geometric, therefore we would need to identify the mathematical quantity $\nabla_\mu R_{\alpha\beta}$ as a physical object, similar to those shown in Equation (5), or to establish a connection so that the geometric object $\nabla_\nu R_{\beta\lambda} - \nabla_\lambda R_{\beta\nu}$ can be related directly to a physical entity. In fact, for the case of the gravitational field, the geometric object $\nabla_\nu R_{\beta\lambda} - \nabla_\lambda R_{\beta\nu}$ can be expressed in terms of the energy-momentum tensor $T_{\alpha\beta}$ if we employ the Einstein field equations of general relativity given in Equation (4). Then the current $J_{\beta\nu\lambda}$ is determined directly by the energy-momentum tensor as $J_{\beta\nu\lambda} = k\left(\nabla_\nu\left(T_{\beta\lambda} - g_{\beta\lambda}T/2\right) - \nabla_\lambda\left(T_{\beta\nu} - g_{\beta\nu}T/2\right)\right)$ and the field equations given in Equation (12) now take the form

$$\nabla_\mu R^\mu_{\beta\nu\lambda} = k\left(\nabla_\nu\left(T_{\beta\lambda} - \frac{1}{2}g_{\beta\lambda}T\right) - \nabla_\lambda\left(T_{\beta\nu} - \frac{1}{2}g_{\beta\nu}T\right)\right) \quad (14)$$

A common energy-momentum tensor that can be used is the energy-momentum tensor for the perfect fluid given as $T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + pg_{\alpha\beta}$. On the other hand, for a pure gravitational field we have $J_{\beta\nu\lambda} = 0$. Using the definition for the current $J_{\beta\nu\lambda}$ given in Equation (13) then we obtain

$$\nabla_\nu R_{\beta\lambda} - \nabla_\lambda R_{\beta\nu} = 0 \quad (15)$$

Using the condition $\nabla_\mu g^{\alpha\beta} = 0$ we arrive at the condition that defines an Einstein manifold

$$R_{\alpha\beta} = kg_{\alpha\beta} \quad (16)$$

where k is a dimensional constant. Also, for the case $J_{\beta\nu\lambda} = 0$, the field equations for the Riemann curvature tensor given in Equation (12) reduces to

$$\nabla_\mu R^\mu_{\beta\nu\lambda} = 0 \quad (17)$$

Again, using the decomposition of the Riemann curvature tensor into the Ricci curvature tensor and the Weyl tensor $C_{\alpha\beta\mu\nu}$, as given in Equation (1), the relation between the Riemann curvature tensor and the metric tensor can be established for three-dimensional differentiable manifolds. Furthermore, as in the case of establishing an energy-momentum tensor of the form $kT_{\alpha\beta} = R_{\alpha\beta} - (1/2)g_{\alpha\beta}R$ from the field equations formulated in terms of the Ricci curvature tensor as given in Equation (5), we speculate that there should also exist an energy-momentum tensor of the type associated with the mathematical object from which the field equations are formulated, and this is the Riemann curvature tensor as given in Equation (12). Therefore, we suggest that the corresponding energy-momentum tensor should be established in the form

$$kT^{\mu}_{\beta\nu\lambda} = R^{\mu}_{\beta\nu\lambda} - \frac{1}{2} g_{\beta\lambda} R^{\mu}_{\nu} \tag{18}$$

The reason for the suggestion is that by contracting the indices μ and ν in Equation (18) we would recover the energy-momentum tensor given in Equation (3) for the Einstein field equations of general relativity. Also, as in the case of Einstein field equations of general relativity, Equation (18) may also be regarded as field equations that could be used to describe the gravitational field as curved spacetime. In the next section we discuss how to reformulate the field equations of general relativity given the form of energy-momentum tensor of the type presented in Equation (18).

3. A Reformulation of the Field Equations of General Relativity

As we have mentioned above, if matter and the spacetime continuum are endowed with intrinsic geometric structures of a differentiable manifold described by the Riemann curvature tensor $R^{\alpha}_{\beta\mu\nu}$ then it seems reasonable to assume that related physical objects should also be described by a physical tensor which has a mathematical formation comparable to the Riemann curvature tensor and therefore should also be expressed in the form $T^{\alpha}_{\beta\mu\nu}$. And, importantly, the physical tensor $T^{\alpha}_{\beta\mu\nu}$ should be required to satisfy the Bianchi identities

$$\nabla_{\lambda} T_{\alpha\beta\mu\nu} + \nabla_{\lambda} T_{\alpha\beta\mu\nu} + \nabla_{\lambda} T_{\alpha\beta\mu\nu} = 0 \tag{19}$$

Then we may assume further that the usual energy-momentum tensor $T_{\alpha\beta}$ can be obtained from the tensor $T^{\alpha}_{\beta\mu\nu}$ by the contraction $T_{\alpha\beta} = T^{\mu}_{\alpha\mu\beta}$. Provided with this assumption then each term of the tensor $T^{\alpha}_{\beta\mu\nu}$ would be identified as the flux of the β component of the momentum through a surface of constant x^{ν} for a pair of value of the indices α and μ. Furthermore, from the Bianchi identities satisfied by the Riemann curvature tensor $R^{\alpha}_{\beta\mu\nu}$ as shown in Equation (10) and the proposed physical tensor $T^{\alpha}_{\beta\mu\nu}$ as shown in Equations (19), we may possibly suggest that they might satisfy the equation

$$R_{\alpha\beta\mu\nu} = kT_{\alpha\beta\mu\nu} \tag{20}$$

where k is a dimensional constant. Even though we can assume that the contracted entity $T_{\alpha\beta} = T^{\mu}_{\alpha\mu\beta}$ can be identified with an energy-momentum tensor, the existence of the presumed physical quantity $T^{\alpha}_{\beta\mu\nu}$ itself remains a speculation even though we may suggest further that it could be related to dark matter. Contracting Equation (19) by $g^{\alpha\mu}$ we obtain

$$\nabla_{\lambda} T_{\beta\nu} - \nabla_{\nu} T_{\beta\lambda} + \nabla_{\mu} T^{\mu}_{\beta\nu\lambda} = 0 \tag{21}$$

Contracting again Equation (11) and Equation (21) using $g^{\beta\nu}$, then the equations obtained from the contractions can be rewritten in the forms

$$\nabla_\beta\left(R^{\alpha\beta}-\frac{1}{2}g^{\alpha\beta}R\right)=0 \tag{22}$$

$$\nabla_\beta\left(T^{\alpha\beta}-\frac{1}{2}g^{\alpha\beta}T\right)=0 \tag{23}$$

With these identities then, similar to the case when formulating Einstein field equations of general relativity, we may assume that the Ricci curvature tensor $R_{\alpha\beta}$ and the energy-momentum tensor $T_{\alpha\beta}$ are related to form a system of field equations of the form

$$R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R=k\left(T_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}T\right) \tag{24}$$

By contraction, we obtain from Equation (24) the relation $R=kT$, therefore the field equations given in Equation (24) can also be written in the simpler form

$$R_{\alpha\beta}=kT_{\alpha\beta} \tag{25}$$

In fact, Equation (25) was the field equations that Einstein proposed in his first attempt to establish relativistic field equations for the gravitational field. However, Einstein abandoned this form of relativistic field equations because the tensor $T_{\alpha\beta}$ itself was required to satisfy the conservation law $\nabla^\beta T_{\alpha\beta}=0$, on the other hand $\nabla^\beta R_{\alpha\beta}\neq 0$.

To illustrate the current formulation of the field equations of general relativity, let us consider the energy-momentum tensor for the perfect fluid given as follows

$$T_{\alpha\beta}(PF)=\left(\rho+\frac{p}{c^2}\right)u_\alpha u_\beta+pg_{\alpha\beta} \tag{26}$$

If the energy-momentum tensor $T_{\alpha\beta}(PF)$ for the perfect fluid satisfies the conservation law $\nabla^\beta T_{\alpha\beta}(PF)=0$ then according to our present formulation for the gravitational field there should exist an energy-momentum tensor $T_{\alpha\beta}$ so that $T_{\alpha\beta}(PF)=T_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}T$, and the tensor $T_{\alpha\beta}$ can be determined from the relation

$$T_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}T=\left(\rho+\frac{p}{c^2}\right)u_\alpha u_\beta+pg_{\alpha\beta} \tag{27}$$

Inversely, if the energy-momentum tensor for the perfect fluid given in Equation (26) is identified with the tensor $T_{\alpha\beta}$ that satisfies the field equations given in Equation (25), $T_{\alpha\beta}(PF)=T_{\alpha\beta}$, then the tensor that satisfies the conservation law should take the form $T_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}T$, where $T=g^{\alpha\beta}T_{\alpha\beta}(PF)=3p-\rho c^2$. In this case, the energy-momentum tensor that satisfies the conservation law $\nabla^\beta\left(T_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}T\right)=0$ is given as

$$T_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}T=\left(\rho+\frac{p}{c^2}\right)u_\alpha u_\beta+\left(-\frac{p}{2}+\frac{\rho c^2}{2}\right)g_{\alpha\beta} \tag{28}$$

4. Conclusion

In this work, we have examined the intrinsic geometric character of the field equations of general relativity and then proposed to formulate relativistic field equations in terms of the Riemann curvature tensor $R^{\alpha}_{\beta\mu\nu}$. The resulted field equations could be applied to the gravitational field when the Ricci curvature tensor $R_{\alpha\beta}$ contained in the field equations is related to the usual energy-momentum tensor $T_{\alpha\beta}$ via Einstein field equations of general relativity. Furthermore, since matter and the spacetime continuum can be perceived to be endowed with intrinsic geometric structures of a differentiable manifold described by the Riemann curvature tensor $R^{\alpha}_{\beta\mu\nu}$, then it seems reasonable to suggest that physical objects should also be described by a physical tensor $T^{\alpha}_{\beta\mu\nu}$ which has a mathematical formation comparable to the Riemann curvature tensor. As a consequence, we can obtain field equations of general relativity in the form $R_{\alpha\beta} = kT_{\alpha\beta}$, with $R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta}$ and $T_{\alpha\beta} = T^{\mu}_{\alpha\mu\beta}$. The relativistic field equations that we have formulated using the Riemann curvature tensor also integrated into the framework that we have presented in our previous works that a classical field can be formulated from a system of field equations written in the general form $\nabla_{\mu} M = J$, where M is a mathematical object, J a physical entity identified from a geometrical object, and ∇_{β} a covariant derivative.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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