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Efficient Pricing of Low Volatility Path Dependent Options

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Abstract

Asian options are generally priced using arithmetic or geometric averages of the underlying stock. However, these methods are not suitable when stock's volatilities are very low. The motivation to develop derivative prices based on averaging the underlying asset stems from the robust features associated with Asian options which suggest that they are more suited to African markets where prices can be dormant for long periods resulting in low volatilities in stock prices. We propose the use of the modal average as the measure of the underlying stock price when stocks have low volatilities instead of the more popular arithmetic and geometric averages. In particular, the stock price is assumed to follow Geometric Brownian Motion and using the concept of maximum of a function, a model for the modal average of the underlying stock is derived. A process of obtaining the price of a call option is subsequently developed. Theoretically, we prove further that for very low volatilities the modal average model is a better estimator of the expected average of the stock price and consequently produces cheaper option prices than geometric and arithmetic average models. Using data from the Ghana Stock Exchange and the Nasdaq, the proposed model is used to price options sold on selected stocks on the exchange. The numerical results consistently show that for underlying stocks with volatility less than 3%, the modal average model provides cheaper call options than the arithmetic or geometric averages pricing models.

Keywords

Average Options, Arithmetic Average, Geometric Average, Modal Average, Volatility

1. Introduction

Average value options or as it is known elsewhere, Asian options, are options whose price is determined based on the average price of the underlying stock or asset. Compared to European options, Asian options are cheaper and better suited to hedging purposes. In addition, they can also reduce the risk of price manipulations especially near the option's maturity date. The use of averages of the underlying asset to price options has seen considerable investigation by researchers and different methods have been suggested to analyze the average. Averaging includes discrete and continuous averages. Currently, no general analytical solution to price arithmetic average option is known and as such several numerical methods have been proposed to obtain arithmetic averages. The problem is that even in cases where analytical or numerical solutions exist, arithmetic or geometric averaging is still unsuitable to obtain the average of the path of the underlying asset especially in cases where stocks have very low volatilities and in many cases averaging by geometric or arithmetic averages usually results in overpricing of the option. Unfortunately, many African stocks (Ghana, Kenya and Nigeria, for example) exhibit low volatilities as a result of the tendency of some stock prices to remain dormant or change marginally for long periods. This study thus examines the use of a modal average as the average for underlying asset in pricing options. The main objective is to develop a new option pricing model based on the modal average of the underlying asset. Specifically, the paper will

- Develop an option pricing model based on the modal average of the underlying asset;
- Use the model to price options on stocks listed on some stock exchanges;
- Compare the option prices obtained from the modal average to option prices obtained using arithmetic and geometric averages models;
- Analytically show that for low volatility assets the modal average model offers cheaper options compared to the arithmetic and geometric averages models.

Literature Review

The mathematical theories underpinning the rigorous treatment of Asian options are well rooted in stochastic calculus, measure theory, martingales and largely partial differential equations. Currently, the averaging the underlying assets of Asian options have primarily been realized through the use of arithmetic and geometric averages. Geometric average options can be priced analytically since the product of lognormal random variables are also lognormally distributed. Thus, explicit closed-form expressions of geometric average options exist and have been derived by Kemna and Vorst [1], Angus [2]. Arithmetic average options however, do not have closed form solutions. This is because in the discrete case, if the binomial model approach is used to price the option, it is necessary to keep track of 2^n possible paths or has cardinality of 2^n, where n is the number of periods. This makes it very difficult to examine the paths if n is large. In the continuous case, if the underlying stock is assumed to have a lognormal distribution, that is, following Geometric Brownian Motion as in Samuelson [3] and Black, Scholes [4], then the arithmetic average does not have a known distribution since the sum of lognormal random variables are not lognormally distributed. For this reason, several approximations methods that produce closed form expressions have been proposed. These methods are based on approximating the underlying asset by some standard probability distribution with known parameters and density. Turnbull and Wakeman [5] used the lognormal distribution of the generalized Edgworth to approximate the underlying asset. Vorst [6] achieved an approximation using adjusted strike price which is given as the difference in expectation of the arithmetic average. Curran [7] conditioned on the geometric average by integrating with respect to its lognormal distribution. In the discrete case where approximations of the underlying are not available, Boyle [8], Broadie and Glasserman [9] have employed various numerical methods. Shreve [10] established a method which included replication and self-financing strategies. Other authors have priced Asian options using Partial Differential Equations (PDEs). This approach involves solving two PDEs; one for the underlying stock and the other for the average stock price. But this approach is unreliable as solving two dimensional PDEs is subject to oscillations. Rogers and Shi [11] reduced the two dimensional problem to one dimensional problem but Vecer [12] showed that the resulting one-dimensional problem is difficult to solve numerically. Semi-analytic techniques have also been developed by Hoogland and Neumann [13] using scale invariance methods. Recent studies have primarily focused on novel computational applications and testing efficiency and speed of convergence of the models. Monte Carlo simulation for instance, has gained prominence and has widely been employed as an effective simulation technique. Rubinstein and Kroese [14], Kechejian *et al.* [15] have examined these methods in pricing arithmetic Asian options. Although the geometric and arithmetic average options have been widely accepted as model approaches in pricing Asian options, the pricing of options on low volatility assets continue to remain problematic. For instance, Geman and Yor [16] derived the Laplace transform for a continuous arithmetic Asian option, but the numerical inversion of the Laplace transform is not suitable when volatility is low. Linesky [17] employed Monte Carlo methods to obtain numerical solutions of arithmetic average options but in low volatility stocks convergence remain slow. Fu et. al [18] showed that there is a problem of slow convergence especially for low volatility stocks.

2. Methodology

2.1. Option Pricing Modeling

The pricing of Asian options is primarily based on the use of expectations in which the price of the option is valued as a replicating portfolio, its value equal to the price of its discounted expected payoff at maturity.

Theorem 1

Let $X(t)$ satisfies the stochastic differential equation

$\mathrm{d}X(t) = rX(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t)$, where $W(t)$ is Wiener process under the measure $\mathbb{Q}$. Let $V(X(t),t)$ be the value of a contingent claim on $X(t)$, then by Itô formula

$$\frac{\partial V(X(t),t)}{\partial t} + r\frac{\partial V(X(t),t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 V(X(t),t)}{\partial x^2} - rV(X(t),t) = 0$$

and $V(X(t),t)$ has the solution

$$V(X(t),t) = \mathbb{E}\left(\mathrm{e}^{-\int_t^T r(X(u),u)\mathrm{d}u} V(X(T),T) \middle| \mathcal{F}_t\right) \tag{1}$$

Pricing Asian Options

Two averages are used to price Asian options: Geometric and Arithmetic averages. Assume that the stock price is given by $S(t)$, then the arithmetic average is given by

$$\mathbb{A} = \frac{1}{T-t}\int_t^T S(u)\mathrm{d}u \tag{2}$$

and the geometric average is given by

$$\mathbb{G} = \exp\left(\frac{1}{T-t}\int_0^T \ln S(u)\mathrm{d}u\right) \tag{3}$$

In the arithmetic case, $\mathbb{A} = \frac{1}{T-t}\int_t^T S(u)\mathrm{d}u$ is not analytically tractable and so numerical methods are employed.

In the geometric case we have Fixed-Strike Geometric Asian call and Floating-Strike Geometric Asian call.

The Fixed-Strike Geometric Asian call option price is given by

$$C(S(t),t) = \mathrm{e}^{r(T-t)}\mathbb{E}_{\mathbb{Q}}\left(\max\left[\mathbb{G}-K,0\right]\Big|_{\mathcal{F}}\right) \tag{4}$$

Similarly, the Floating-Strike Geometric Asian call option has payoff

$$C(S(t),t) = \mathrm{e}^{r(T-t)}\mathbb{E}_{\mathbb{Q}}\left(\max\left[S(T)-\mathbb{G},0\right]\Big|_{\mathcal{F}}\right) \tag{5}$$

2.2. The Modal Average Option Price

In this paper a systematic algorithm is derived to price an option from the premise of probability spaces using a stock price as the underlying asset. The fundamental view is to derive the option price based on an underlying stock whose price movement is captured as a realization of the price event in a measurable space, Intuitively, the measurable space admits a level of uncertainty in an economy equipped with a filtration, viewed as information available at time t and on which is defined a probability measure. Throughout the text we will assume an efficient market with continuous trading, more specifically, we assume that our model is applied in the Black-Scholes world. Thus, we are basically concerned with an European style option whose underlying stock is averaged over a specified time period. In particular, we will assume the stock price to follow a stochastic process and modeled by the Geometric Brownian motion. In developing the modal average model, we use the concept of maximum of a function we derive a method to average of the underlying stock and proceed to test the model using data from GSE and the Nasdaq. We further show analytically that for low volatility stocks the derived model based on modal average produces cheaper option when compared to the arithmetic and geometric average models.

2.3. The Stock Price Process

Consider a stock with prices $\omega_0, \omega_1, \cdots, \omega_n$ at times $0 = t_0 < t_1 < \cdots < t_n = T$. Let $f_0, f_1, \cdots, f_n$ denote the frequencies of $\omega_0, \omega_1, \cdots, \omega_n$. Define Ω such that $\Omega = \{f_0, f_1, \cdots, f_n\}$. Let A be a subset of Ω and let $\mathcal{F}$ denote a collection of subsets of $\omega_i' s$ on Ω. Then surely $\mathcal{F}$ forms a σ-algebra of frequencies $f_i' s$ on Ω with measure space given by $(\Omega, \mathcal{F})$. Let $\mathbb{P}$ denote a probability measure defined on this space, then $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space for the frequency of stock prices. To develop the continuous process let ω be the realization of a stock price on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $X(\omega)$ be a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X : \Omega \to \mathbb{R}$, then $X(\omega)$ is a measurable function on $(\Omega, \mathcal{F}, \mathbb{P})$. By considering the frequencies of $X(\omega)$ we are now interested in some function of $X(\omega)$. The distribution of the frequency of $X(\omega)$ is thus a composite of some function f and $X(\omega)$, where $f : \mathbb{R} \to \mathbb{R}$. Let's denote this as $f(X(\omega))$. Since the pre-images $f^{-1}(X(\omega)) \in (\Omega, \mathcal{F}, \mathbb{P})$, it follows that $f(X(\omega))$ are measurable in $(\Omega, \mathcal{F}, \mathbb{P})$ and consequently $f(X(\omega))$ are random variables. If there are n random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$, it follows that $f(X_1(\omega), X_2(\omega), \cdots, X_n(\omega))$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Thus, we have some random variables from some probability space being mapped onto some vector space $\mathbb{R}^n$, such that for any Borel set $\mathcal{B} \in \mathcal{B}(\mathbb{R}^n)$ the preimages $f^{-1}(X_1, X_2, \cdots, X_n) \in (\Omega, \mathcal{F}, \mathbb{P})$. The frequency distribution of X is now a mapping of X from $\mathbb{R}^n$ onto some vector space $\mathbb{R}^n$. Since f is Borel measurable it follows that $f(X_1, X_2, \cdots, X_n)$ is a random variable.

Figure 1 shows the random variable $X(\omega)$ from some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ mapped onto some vector space $\mathbb{R}$. The pre-images of $X(\omega)$ of some Borel sets in $\mathbb{R}$ are measurable in $(\Omega, \mathcal{F}, \mathbb{P})$. The function f maps $X(\omega)$ onto some space in another space $\mathbb{R}$. Let f be a measurable function in $(\Omega, \mathcal{F}, \mathbb{P})$, then the pre-images of $f(X(\omega))$ are also measurable in $(\Omega, \mathcal{F}, \mathbb{P})$. For this study we are interested in the value of $x \in \mathbb{R}$ corresponding to the maximum frequency. Let $X_1, X_2, \cdots, X_n$ be random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{P}(x_1), \mathbb{P}(x_2), \cdots, \mathbb{P}(x_n)$ be the frequency (probability) distribution of $X_1, X_2, \cdots, X_n$ then the function $f : \mathbb{R} \to \mathbb{R}$ represents the probability distribution of $X_1, X_2, \cdots, X_n$. The mode of this distribution is the value of X corresponding to the maximum of the probabilities in the distribution. If $X = x_i$ represent the stock price and $\mathbb{P}(x_i)$ represents frequency of occurrence of x_i or the probability that x_i occurs, then we seek for the numerical value of X corresponding to $\mathbb{P}_{\max}(x_i)$.

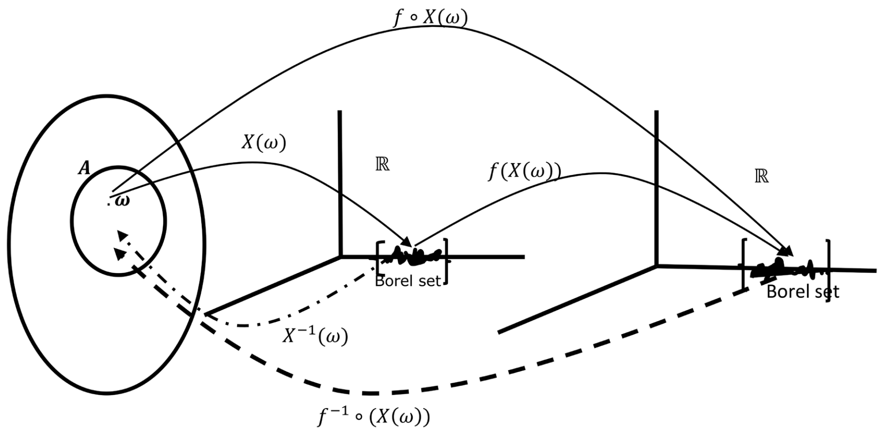

Figure 1. Spatial distribution of the stock price process.

This approach explains the maximum frequency when stock price events are discrete. Now in the event of increase in trading activity the number of steps of the discrete process increases and the process can generally be approximated by a continuous process. In limiting function of the discrete activities described by the random walk process is now inherited by the Brownian motion process which then assumes all the properties of the random walk with increments normally distributed with mean zero and variance *t*.

2.3.1. Mode of a Random Variable

Let $X_i \geq 0$ be a random variable. Let $\mathbb{P}(X_i) \geq 0$ be the frequency distribution of X_i then function $f: \mathbb{R} \to \mathbb{R}$ represents the probability distribution of X_i. The mode of this distribution is the value of X on the real line $\mathbb{R}$ corresponding to the maximum frequency of the distribution. If X represents the stock price and $\mathbb{P}(X_i)$ represent the frequency that the stock will assume the price X_i then we seek for the value of X that corresponds to the maximum frequency.

2.3.2. Mode of a Discrete Probability Distribution Function

Let $X_1, X_2, \cdots, X_n$ be random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{P}(x_1), \mathbb{P}(x_2), \cdots, \mathbb{P}(x_n)$ be the probability distribution of $X_1, X_2, \cdots, X_n$ then the function $f: \mathbb{R} \to \mathbb{R}$ represents the joint probability distribution of $X_1, X_2, \cdots, X_n$. The mode of this distribution is the value of X corresponding to the maximum of the probabilities in the distribution. If X represents the stock price and $\mathbb{P}(x_1)$ frequency of occurrence of x or the probability that x occurs, then we seek for the numerical value of X corresponding to maximum $\mathbb{P}$.

2.3.3. Mode of a Continuous Probability Distribution Function

Let $\mathbb{P}(x)$ be the function that assigns values to the distribution of a random variable X_i such that $\mathbb{P}(X_i) = \mathbb{P}^{\max}$ where $\mathbb{P}^{\max}$ is the maximum of the probability distribution of the random variable. X_i is the mode of $\mathbb{P}(x)$. That is, $\{X_{\text{mode}} = x$ such that $\mathbb{P}^{\max}$ is the maximum probability$\}$. If $f(x)$ is a probability distribution function then the maximum of $f(x)$ is given by finding the stationary points of $f(x)$, conditioned by $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} < 0$.

Theorem 2

Let X be a continuous random variable such that $f(x)$ is the probability density function of X. Suppose $f(x)$ is smooth enough such that the first and second derivative exists. Let x be the value of X that maximizes $f(x)$. That is, the value of x at $\frac{\mathrm{d}f}{\mathrm{d}x} = 0$. If $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} < 0$ then $X = x$ is the mode of the distribution function $f(x)$ and we write

$$\bar{X}_M = \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{\bar{X}_M} = 0 : \left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{\bar{X}_M} < 0 \tag{6}$$

or

$$\bar{X}_M = \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{\bar{X}_M} = 0\Big|_{\left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{\bar{X}_M} < 0} \tag{7}$$

where $\bar{X}_M$ is the mode of $f(x)$.

2.3.4. Numerical Algorithm to Determine the Mode

To determine the maximum of $f(x)$, numerical methods are employed. In effect, the mode is the value of the random variable X corresponding to the maximum of the $f(x)$, provided the $f(x)$ satisfies the axioms of a probability density function and smooth enough such that the first and second derivatives exist. The problem thus reduces to solving the optimization problem of maximize $f(x)$ on $[a,b]$. That is

$$\max f(x)$$

subject to $a \le x \le b$

If the first and second derivatives of $f(x)$ exists and is continuous on $[a,b]$ then it can be to solved as follows:

- Compute all distinct zeros of $f'(x)$ in the interior of the interval $[a,b]$;
- Evaluate $f(x)$ at zeros and at the endpoints a and b;
- Test if $f''(x) < 0$.

Other numerical approximation methods can be used to determine the maximum for both continuous as well as the discrete case. A typical example is the Golden Section Search Optimization Method that searches for the extremum (minimum of maximum) of a strictly unimodal function. The algorithm is the limit of the Fibonacci search.

2.3.5. The Equation of the Modal Average Option Price

Let $\bar{X}_M$ be the modal value of the stock price with diffusion

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t)$$

in a given time interval $[0,T]$. Under the measure $\mathbb{P}$ the discounted call option price is

$$C\left(S\left(t\right),t\right)=\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\int_t^T\mu\mathrm{d}s\right)\left[\max\left(\bar{X}_M-K,0\right)\Big|_{\mathcal{F}_t}\right]\right] \tag{8}$$

But $\bar{X}_M:\frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{\bar{X}_M}=0\Big|_{\frac{\mathrm{d}^2f}{\mathrm{d}x^2}\Big|_{\bar{X}_M}<0}$ and so the price of a call option on $S\left(t\right)$ is given by

$$C\left(S\left(t\right),t\right)=\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\int_t^T\mu\mathrm{d}s\right)\left[\max\left(\bar{X}_M:\frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{\bar{X}_M}=0\Big|_{\frac{\mathrm{d}^2f}{\mathrm{d}x^2}\Big|_{\bar{X}_M}<0}-K\right)\Big|_{\mathcal{F}_t}\right]\right] \tag{9}$$

Given an indicator random variable $\mathbb{I}_{\mathbb{A}}$ defined as

$$\mathbb{I}_{\mathbb{A}}=\begin{cases}1, \text{ if } \bar{X}_M:\frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{\bar{X}_M}=0\Big|_{\frac{\mathrm{d}^2f}{\mathrm{d}x^2}\Big|_{\bar{X}_M}<0}>K\\ 0, \text{ if } \bar{X}_M:\frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{\bar{X}_M}=0\Big|_{\frac{\mathrm{d}^2f}{\mathrm{d}x^2}\Big|_{\bar{X}_M}<0}\le K\end{cases}$$

Then the condition to exercise the option is

$$C\left(S\left(t\right),t\right)=\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\int_t^T\mu\mathrm{d}s\right)\left[\max\left(\bar{X}_M:\frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{\bar{X}_M}=0\Big|_{\frac{\mathrm{d}^2f}{\mathrm{d}x^2}\Big|_{\bar{X}_M}<0}-K,0\right)\Big|_{\mathcal{F}_t}\right]\times\mathbb{I}_{\mathbb{A}}\right] \tag{10}$$

3. Proof That the Modal Average Is the Most Efficient Pricing Model for Low Volatility Stocks

The modal average has been proposed as a measure of the average of the underlying asset and we will now proceed to prove that for low volatility stocks, the modal average is indeed a better estimator of the average stock than the arithmetic or geometric averages. This is achieved by proving that, in theory, for sufficiently small value of x, the mode is a better estimator of the expected value (stock mean) than geometric or arithmetic averages.

Theorem 3

Let $f\left(x\right)$ be a probability function of a random variable X whose maximum value is given as $f\left(\xi_M\right)$, where ξ_M is the random variable defining the maximum of $f\left(x\right)$ at $X=x$. Then for a sufficiently small Δx the modal average is the best estimate of the expected value of $f\left(x\right)$, for all $x>0$

3.1. Proof

Consider a probability density function $f\left(x\right)$ which describes the graph in **Figure 2**.

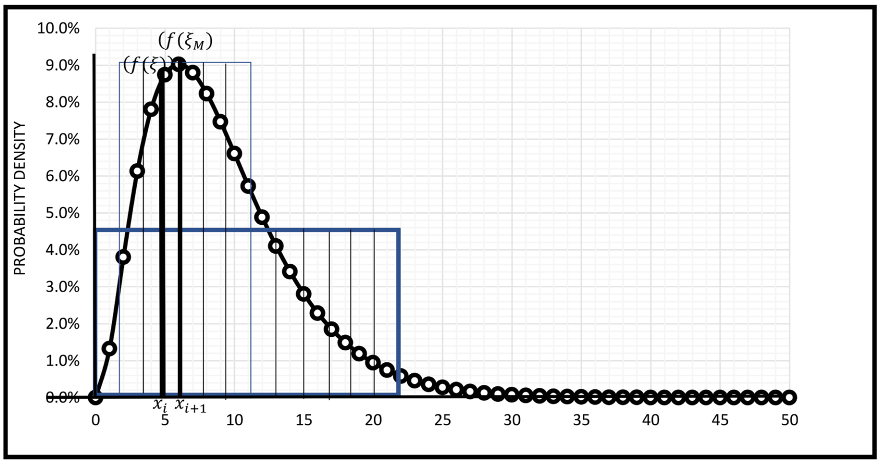

Figure 2. Probability density function of the random process $f(x)$.

Suppose there are n partitions of the interval $[a,b]$. By construction the upper and lower Riemann sums, denoted by U_n and L_n respectively are defined as

$$U_n(f)=\sum_{i=1}^{n}\left(\sup_{x_{i+1,i}} f(x)\right)\Delta x_i$$

$$L_n(f)=\sum_{i=1}^{n}\left(\inf_{x_{i+1,i}} f(x)\right)\Delta x_i$$

By Riemann theorem, as n increases in a manner such that each Δx_i decreases to zero, it can be seen that L_n is monotone increasing, while U_n is monotone decreasing. So, as $n\to\infty$ it follows that L_n and U_n will both converge and f is integrable if and only if

$$\lim_{n\to\infty} U_n f(x)=\lim_{n\to\infty} L_n f(x)=\int_a^b f(x)\mathrm{d}x$$

By the Mean Value Theorem (MVT)

$$f(x)=\frac{\int_a^b f(x)\mathrm{d}x}{\Delta x}=\frac{\text{Area under the curve}}{\Delta x}$$

where $\Delta x=x_{i+1}-x_i$

$$f(x)\Delta x=\text{Area under the curve}$$

Let ξ be a random variable and a function of x such that $\xi\in[a,b]$, then by the Mean Value Theorem

$$\Delta x f(\xi)=\int_a^b f(\xi)\mathrm{d}x$$

Now let $\xi_M\in[a,b]$ such that $\xi_M\geq\xi$ for all $\xi\in[a,b]$, then by the Extreme Value Theorem ξ_M is the value of x corresponding to the maximum value of $f(x)$. Thus, for any small interval Δx the area under the curve estimated using ξ and ξ_M is such that

$$\Delta x f(\xi)\leq\Delta x f(\xi_M)$$

If $f(x)$ is a pdf then the maximum area under the curve equals 1 and hence by the squeeze theorem

$$0 \le \Delta x f(\xi) \le \Delta x f(\xi_M) \le 1$$

The deduction here is that for any small interval Δx the distribution of the area under the curve is a probability density function and its maximum at any point on the curve cannot exceed that at the apex of the curve.

In estimating the mean by the arithmetic average, the area under the curve is obtained using $\xi_i' s$, and the area under the curve is obtained as $\Delta x f(\xi)$. Similarly, the area under the curve when estimating by the modal average is $\Delta x f(\xi_M)$. It therefore follows from the squeeze theorem that in the limit as $\Delta x \to 0$, the area under the curve approaches the maximum 1 only if $\xi_i \to \xi_M$ and $f(\xi_i) \to f(\xi_M)$. Thus, we can write that

$$\lim_{\Delta x \to 0} f(\xi) = f(\xi_M)$$

Hence for sufficiently small Δx the area under the curve $\Delta x f(\xi)$, corresponding to the mean value of the probability density function is bounded by $\Delta x f(\xi_M)$. It must be noted that this situation is possible only for a small value of Δx, as Δx becomes large the area under the curve increases and collapses into a uniform distribution. ■

Let the stock volatility σ be the function of the random variable $\xi \in [0,100]$, then from Theorem 3 we can conclude that for sufficiently low volatility, the best estimate of the expected value of the stock price since $\Delta x \sigma(\xi_M) > \Delta x \sigma(\xi)$ for all $\xi > 0$. In this case the modal average is the best estimate of the expected price of the stock. However, as Δx increases the area under the curve no longer has a maximum and the curve in **Figure 1** collapses into a uniform distribution. In this case the arithmetic and geometric averages are the best estimates of the expected value of the stock price. We confirm this in the simulation process in 3.2.

3.2. Numerical Results

To simulate the sample path of the stock we discretize the path of the stock price by generating sample paths $S(t_i), \cdots, S(t_n)$ at fixed times $t_i, \cdots, t_n$. The procedure begins by assuming the stock price follows the GBM so that

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}B(t) \tag{11}$$

In a short time period Δt the change in Brownian motion $\Delta B(t) = \xi(t)\sqrt{\Delta t}$ where $\xi(t) \sim N(0,1)$.
Hence

$$\Delta S(t) = rS(t)\Delta t + \sigma S(t)\xi(t)\sqrt{\Delta t} \tag{12}$$

$$\frac{\Delta S(t)}{S(t)} = r\Delta t + \sigma \xi(t)\sqrt{\Delta t}$$

Thus, the percentage rate of return of the stock price is normally distributed with mean $r\Delta t$ and variance $\sigma^2 \Delta t$. That is,

$$\frac{\Delta S(t)}{S(t)} \sim N\left(r\Delta t, \sigma^2 \Delta t\right)$$

Now given a function $f\left(S(t),t\right)$, we know by Ito process that

$$\mathrm{d}f\left(S(t),t\right) = \left(rS(t)\frac{\partial f\left(S(t),t\right)}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 f\left(S(t),t\right)}{\partial S(t)^2} + \frac{\partial f\left(S(t),t\right)}{\partial t}\right)\mathrm{d}t + \left(\sigma S(t)\frac{\partial f\left(S(t),t\right)}{\partial S(t)}\right)\mathrm{d}B(t)$$

Applying the process to a stock price with $f = \ln S(t)$ gives

$$\mathrm{d}\ln S(t) = \left(\frac{rS(t)}{S(t)} + 0 + \frac{1}{2}\left(-\frac{1}{S(t)^2}\right)\sigma^2 S(t)^2\right)\mathrm{d}t + \left(\frac{\sigma S(t)}{S(t)}\right)\mathrm{d}B(t)$$

$$\mathrm{d}\ln S(t) = \left(r - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma \mathrm{d}B(t)$$

In discrete time this becomes $\Delta \ln S(t) = \left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\xi(t)\sqrt{\Delta t}$

This gives $\ln\left(S(t)+\Delta t\right) - \ln S(t) = \left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\xi(t)\sqrt{\Delta t}$

$$S(t+\Delta t) = S(t)\left\{\exp\left[r\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\xi(t)\sqrt{\Delta t}\right]\right\} \tag{13}$$

Equation (13) is the path constructing formula for the Monte Carlo simulation of the stock price. For the modal average we will simulate the paths of $S(t_i),\cdots,S(t_n)$ and then compute the value of x corresponding to $\max\{f_0, f_1,\cdots, f_n\}$, where f_i denote the frequency of the stock prices at times $t_i,\cdots,t_n$. The arithmetic and geometric averages are obtained similarly using Equations (2) and (3) respectively.

The Option Price

After simulating the path of the stock price, the average stock price (S_{avg}) is determined as either arithmetic, geometric or modal average. We proceed to price a 3-month Asian call option using the modal, arithmetic or geometric average of the stock price as the underlying asset. Whichever average is used we obtain large samples of option prices $C\left(S(t),t\right)$'s at times $t_i,\cdots,t_n$ and determine the average.

$$C\left(S(t),t\right) = \frac{C_1\left(S(t),t\right) + C_2\left(S(t),t\right) + \cdots + C_n\left(S(t),t\right)}{n}$$

$$C\left(S(t),t\right) = \frac{1}{n}\sum_{i=1}^{n} C_i\left(S(t),t\right) \tag{14}$$

Appendix A shows the simulated stock prices and corresponding option prices computed for stocks listed on the Ghana Stock exchange and on the Nasdaq using geometric, arithmetic and modal averages.

Figures 3-6 show the graphs of modal average option prices against arithmetic and geometric average option prices on GSE and Nasdaq.

4. Discussion of Results

Figure 3 and **Figure 4** shows the plots of option price of stocks against volatilities of stocks on GSE and the Nasdaq when the underlying stock is averaged using the arithmetic and modal averages. **Figure 3** shows that on GSE, for stocks with volatilities between 0% to 3%, the graph of the modal average options prices consistently lies below the graph of options prices of arithmetic average. This

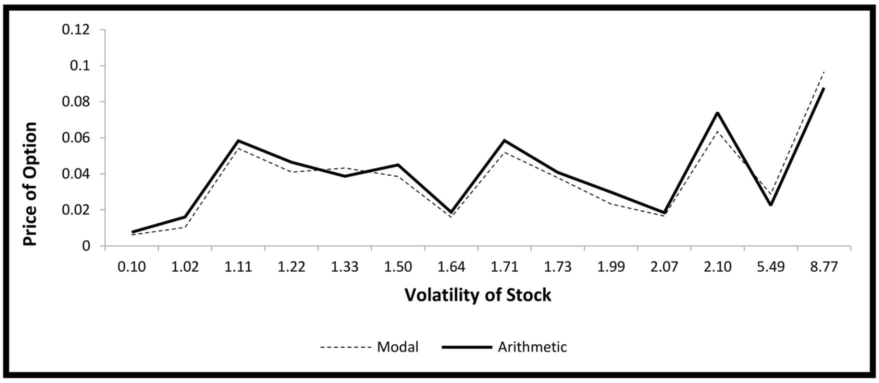

Figure 3. Comparison of Modal average option prices and Arithmetic average options prices on Ghana Stock Exchange.

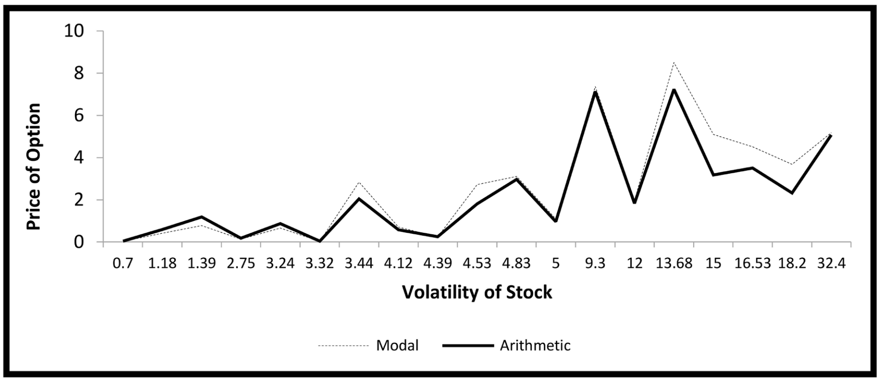

Figure 4. Comparison of Modal average option prices and Arithmetic average options prices on Nasdaq.

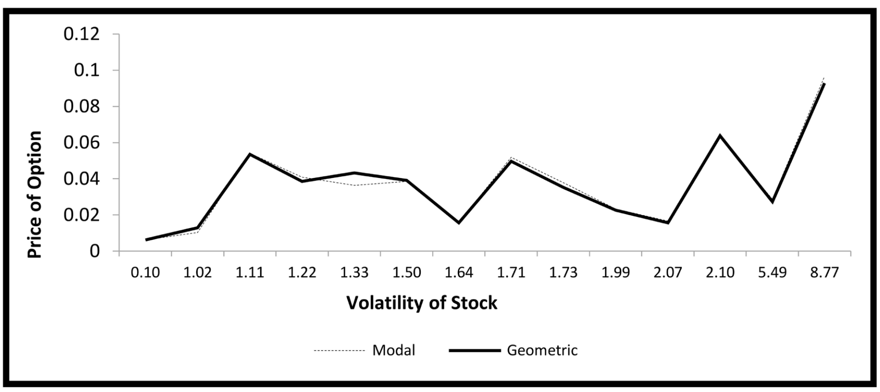

Figure 5. Comparison of Modal average option prices and Geometric average options prices on Ghana Stock Exchange.

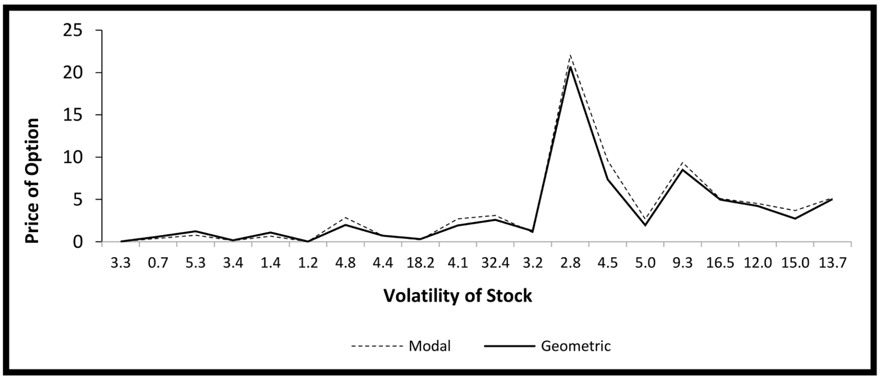

Figure 6. Comparison of Modal average option prices and Geometric average options prices on Nasdaq.

means that the arithmetic average overprices the option when volatility is less than 3%. On the other hand, when volatility is above 3%, we realise that the graph of the modal average options prices consistently lies above the graph of options prices of arithmetic average

Figure 4 also shows with data from Nasdaq, if volatility is less than 3%, the graph of the option price using the modal average model consistently lies below that of the arithmetic average model graph. Similarly, on Nasdaq if volatility is more than 3% the graph of the options prices for the modal average model is always lies above the arithmetic averaged options graph.

Figure 5 and Figure 6 also shows the plots of option prices against volatilities of stocks from GSE and Nasdaq respectively for modal and geometric averages. We realise that when modal average is used, the graph of the option prices consistently lies below that of the geometric averaged option prices when stock volatility is less than 3%. However, above 3% the graph of geometric average option prices always lies below the graph of the modal average option price although the price differences are not significant.

The results show that:

- For stocks with volatilities less than 3%, the modal average model gives lower option prices than the arithmetic and geometric average models on both GSE and the Nasdaq. This shows that for stocks with volatility of less than 3%, is more accurate as the arithmetic and geometric average models overprice the option.
- For stocks with volatilities between 3% and 5%, the modal average model still gives slightly lower option prices than the arithmetic average model. However, the difference in option prices using the modal average model is not significantly different from the prices obtained by using geometric average prices.
- For stock volatility greater than 5%, the arithmetic and geometric average models give lower option prices or are more accurate than the modal average model. In essence, the modal average model overprices the option when volatility is greater than 5%.

The numerical results presented here validates the theoretical assertion that for stocks with very low volatilities, that is, less than 3%, averaging over the life of the option using the modal average of the stock produces a more accurate price for the option.

5. Conclusion

This study examined the use of the modal average of the underlying asset in pricing an Asian option. The study introduced a new model and established a new mathematical framework for the valuation of options with the modal average stock price. We made the assertion that for low volatility options the modal average model is accurate in pricing options when compared to models using geometric and arithmetic averages. In furtherance, an analytical prove is developed to establish this claim. Numerical confirmation to this assertion is advanced using Monte Carlo simulations. It is established that the modal average model gives an accurate price and suitable for pricing options in the case where the underlying stock has very low volatilities ranging from 0% to 3%. This approach would be particularly useful for the valuation of options in low volatility regimes such as on Ghana Stock Exchange and many other African stock exchanges.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix A

Table 1. Simulated stock prices using Geometric, Arithmetic and Modal Averages on GSE.

	Stock Parameter		Average Stock Price		
Stock/Equity	Strike Price	Volatility	Geometric	Arithmetic	Modal
Benso	4.10	1.50	4.1394	4.1385	4.1385
CAL Bank	1.02	1.64	1.0256	1.0259	1.0259
Ecobank Bank GH	7.70	1.11	7.6535	7.654	7.654
Enterprise GP	1.75	1.99	1.7727	1.7733	1.7733
Fan Milk	5.25	1.22	5.2885	5.2909	5.2909
GCB	5.30	1.71	5.3489	5.3519	5.3519
Guinness	3.20	1.73	3.2353	3.2379	3.2379
GOIL	1.05	0.10	1.0562	1.0562	1.0562
Golden Star	2.34	1.02	2.3529	2.3503	2.3503
HFC	1.50	2.07	1.5157	1.5166	1.5166
Standard Chartered	20.35	1.00	20.4685	20.4809	20.4809
Societe General	1.00	5.49	1.0273	1.0225	1.0287
TOTAL GH.	6.10	2.10	6.1638	6.1634	6.1634
Tallow Oil	34.99	2.10	35.3911	35.356	35.356
Unilever	10.70	8.77	10.7929	10.7877	10.7877
Mega Africa	4.50	1.33	4.5363	4.5432	4.5432

Table 2. Option prices on GSE using geometric, arithmetic and modal average stock prices.

	Stock Parameter		Option Price		
Stock/Equity	Strike Price	Volatility	Geometric	Arithmetic	Modal
Benso	4.10	1.50	0.0391	0.0449	0.0385
CAL Bank	1.02	1.64	0.0156	0.0187	0.0159
Ecobank GH	7.70	1.11	0.0535	0.0583	0.0540
Enterprise GP	1.75	1.99	0.0227	0.0298	0.0233
Fan Milk	5.25	1.22	0.0385	0.0464	0.0409
GCB	5.30	1.71	0.0497	0.0585	0.0519
Guinness	3.20	1.73	0.0353	0.0408	0.0379
GOIL	1.05	0.10	0.0062	0.0076	0.0062
Golden Star	2.34	1.02	0.0129	0.0161	0.0103
HFC	1.50	2.07	0.0157	0.0185	0.0166
Standard Chartered	20.35	1.00	0.1185	0.1345	0.1309
Societe General	1.00	5.49	0.0273	0.0225	0.0287
TOTAL Ghana	6.10	2.10	0.0638	0.0740	0.0634
Tallow Oil	34.99	2.10	0.4011	0.3801	0.3660
Unilever	10.70	8.77	0.0929	0.0877	0.0966
Mega Africa	4.50	1.33	0.0363	0.0386	0.0432

Table 3. Simulated stock prices using Geometric, Arithmetic and Modal Averages on NASDAQ.

	Stock parameter		Average Stock Price		
Stock/Equity	**Strike Price**	**Volatility**	**Geometric**	**Arithmetic**	**Modal**
Barclays	2.68	3.32	2.7120	2.7167	2.7136
1347PIH	7.80	0.70	7.8418	7.8416	7.8344
Amazon	749.87	5.33	770.5815	771.8901	768.0482
Apple	115.82	3.44	117.7973	117.8611	118.6621
AT&T	42.53	1.39	43.7633	43.7203	43.3087
BCOM	20.18	1.18	20.7807	20.7763	20.5957
Facebook	115.05	4.83	117.6367	118.0125	118.1533
Ford Motors	12.13	4.39	12.4536	12.3851	12.3730
General Electric	31.60	18.20	34.3291	35.2814	33.9188
General Motors	34.84	4.12	35.5684	35.5536	35.4175
Intel Corp	36.27	32.40	41.3124	41.3478	41.4441
Microsoft	62.14	3.24	63.2238	63.0061	62.8056
NY Times	13.30	2.75	13.4765	13.4862	13.4398
ODML	85.79	4.53	87.7146	88.5126	87.6008
Starbucks	55.52	5.00	56.7970	56.4879	56.6176
Tesla	213.69	9.30	223.2820	220.8227	221.0606
Verizon	51.40	16.53	54.8668	54.7580	55.3195
American Airlines	51.40	12.00	53.7148	53.3479	53.7916
Airbus	66.25	15.00	71.2220	69.4272	71.3456
Boeing	155.68	13.68	165.0347	162.9145	164.1845

Table 4. Option prices on NASDAQ using Geometric, Arithmetic and Modal average stock prices.

Stock Parameter			Option Price		
Stock/Equity	**Strike Price**	**Volatility**	**Geometric**	**Arithmetic**	**Modal**
Barclays	2.68	3.32	0.03202	0.03674	0.03360
1347PIH	7.80	0.70	0.04176	0.04155	0.03444
Amazon	749.87	5.33	20.70967	18.17659	22.01807
Apple	115.82	3.44	1.97717	2.04097	2.83951
AT&T	42.53	1.39	1.23223	1.18919	0.77802
BCOM	20.18	1.18	0.60016	0.59578	0.41535
Facebook	115.05	4.83	2.58647	2.96227	3.10304
Ford Motors	12.13	4.39	0.32357	0.25509	0.24300
General Electric	31.60	18.20	2.72882	2.31861	3.68109
General Motors	34.84	4.12	0.72830	0.57745	0.71351

Continued

Intel Corp	36.27	32.40	5.04190	5.07738	5.17368
Microsoft	62.14	3.24	1.08371	0.86606	0.66550
NY Times	13.30	2.75	0.17648	0.18617	0.13975
ODML	85.79	4.53	1.92443	1.80971	2.72019
Starbucks	55.52	5.00	1.27686	0.96778	1.09748
Tesla	213.69	9.30	9.59110	7.13202	9.70333
Verizon	51.40	16.53	4.23520	3.50626	4.51593
American Airlines	51.40	12.00	2.69403	1.82698	2.82816
Airbus	66.25	15.00	4.97160	3.17687	5.09510
Boeing	155.68	13.68	9.35389	7.23382	9.88606