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Laws of Large Numbers for Dynamic Coherent Risk Measures

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Abstract

In this paper, we study the asymptotic behavior of dynamic coherent risk measures in general settings regardless of specific representations of the risk measures. In particular, we develop three different types laws of large numbers (LLN) for the average values of portfolios. These LLNs capture the limiting behavior of time-consistent dynamic coherent risk measures under appropriate conditions. Our results apply to general probability spaces with a sequence of financial returns characterized by a set of probability measures. We show that the limit of these averages will generally be multivalued within an identified set. We give examples to illustrate the potential applicability of our results and derive asymptotic results on estimation for the risk of returns of financial assets using a time-consistent dynamic coherent risk measure induced by a class of g-expectations.

Keywords

Dynamic Coherent Risk Measures, Time Consistency, Law of Large Numbers

1. Introduction

Risk measures are important for investments and many other decision-makings in economics and finance. There is a long history of measuring and managing risk, and various risk management tools have been developed in recent decades. Together with the development of economic and financial theory related to risk and uncertainty, various concepts, quantitative models and estimators to quantify risk have been proposed and studied. These include models and methods for estimating Value-at-Risk, Expected Shortfall, volatility, probability of default, exposure at default and loss given default, etc.

Among these existing risk measures, variances, the Value-at-Risk (VaR) and the Expected Shortfall (ES) are arguably the most popular risk measures in practice. The VaR is defined as the loss in market value of a security over a given time horizon that is exceeded with probability τ, where τ is often set at a small number, say 0.01 or 0.05. However, despite of its popularity, VaR as a risk measure has also been criticized. An important criticism to VaR is that it is not a "coherent" risk measure.

Following the axiomatic approach, Artzner *et al.* [1] define a **coherent** risk measurement from a regulator's point of view.

Definition 1.1 *A mapping* $\rho = \rho_0 : \chi \to \mathbb{R} \cup \{+\infty\}$ *is called a* ***coherent*** *measurement of risk if it satisfies the following conditions for all* $X, Y \in \chi$.

- *Monotonicity: if* $X \le Y$*, then* $\rho(X) \ge \rho(Y)$.
- *Translation Invariance: if* $a \in \mathbb{R}$*, then* $\rho(X+\alpha) = \rho(X) - \alpha$.
- *Positive Homogeneity: if* $\lambda \ge 0$*, then* $\rho(\lambda X) = \lambda \rho(X)$.
- *Sub-additivity:* $\rho(X+Y) \le \rho(X) + \rho(Y)$.

Their starting point is that although we all have an intuitive sense of what financial risk entails, it is difficult to give a good assessment of financial risk unless we specify what a measurement of financial risk actually means. The four axioms have clear meaning in the finance context: Monotonicity says the risk increases when the return decreases; a risk measurement satisfies the translation invariance axiom if adding alpha dollars of capital to an asset reduces the risk measure by alpha dollars; Positive Homogeneity says that the risk exposure of a financial position grows in a linear way as the size of the position increases; Sub-additivity specifies that the risk of the sum never exceeds the sum of the risks, which closely related to the concept of risk diversification in a portfolio of risky assets.

According to the definition of Artzner *et al.* [1], both the standard deviation (variance) and the VaR are not coherent risk measurements (CRM). However, while volatility and VaR don't fall into the category of coherent risk measurement, Expected Shortfall (ES), a coherent risk measurement, has been suggested as an alternative (remedy) for VaR based risk measurement. ES is defined as the expected loss exceeding VaR. These risk measures are mandated in many current regulatory contexts.

Risk measurement and management have returned to the top of the agenda in the wake of the subprime meltdown in 2007-2008. Quite clearly, risk management systems failed to deliver enough information during the recent crisis, and the price paid by the global economy has been heavy. It is evident that government administrators, financial institutions and companies all need robust risk measures and integrated risk management framework that can inspire the confidence of people and prevent extreme events.

In practice, existing methods assume that the underlying distribution of data can be characterized by certain class of probability measures, and there exists a true model that can be consistently estimated. Consequently, risk measures are obtained based on the estimated model.

An important practical issue in economics and finance that has attracted a growing amount of research attention in recent years is model uncertainty—the uncertainty of data generating process. In recent years, researchers demonstrated cumulated evidence that many economic decisions and activities often confront model uncertainty or model misspecification. For example, Chen, Hansen and Hansen [2] argue that in many GMM models, there is potentially a very large set of subjective beliefs/distributions for which the moment conditions will be satisfied.

Model uncertainty or model ambiguity provides additional challenge when we consider risk under such circumstances. Delbaen [3] provides a representation of coherent risk measures in general probability spaces. The definition and early study of such risk measures are static. In practice, financial returns are usually observed over time. As time goes by, changes are made to the position and new information is released. On the next period, the decision-maker wishes to reconsider the risk of its changed position taking into account the new information in a proper way. For this reason, there has been an increasing interest in extending these static risk measures to dynamic environments in recent years.

In this paper, we study dynamic risk measures that can accommodate model uncertainty. The risk measures that we consider in this paper are coherent and can be constructed based on general probability spaces which allows for a set of probability measures that characterizes the stochastic behavior of the economic activity. We develop laws of large numbers for dynamic coherent risk measures. These limiting results provide tools to analyze the risk of average of market values of portfolios. The asymptotic results represent a recognition of the potential effects of the complexities in real-world phenomenon to produce a robust analysis on statistical estimators based on sample averages.

Arguably one of the most fundamental forms of a statistic in practice is the sample average—many statistical estimators can be constructed in the form of a sample average of random variables or transformations of them. The dynamic coherent risk measures of the portfolios can be estimated based on the sample average. For this reason, we focus our attention on the limiting behavior of sample average.

When we study risk measures in a dynamic setting, it is important not to contradict oneself over time in one's risk assessments. This basic idea is summarized as the property of dynamic consistency in the literature. In this paper, we develop three different laws of large numbers for time-consistent dynamic coherent risk measures. The resulting laws of large numbers show that the limit of average returns of portfolios over time will generally be multivalued, with their limit point confined in a deterministic set. We give conditions that parallel to those used in conventional laws of large numbers and provide a robust analysis that works relatively well regardless of the specific representation of dynamic coherent risk measures.

Our result provides some theoretical foundation that can be applied to many research problems in economic and financial applications. To highlight the importance of our results, we study two important examples of time-consistent dynamic coherent risk measures and apply our asymptotic results to evaluate the risk of financial assets. In particular, we study dynamic coherent risk measures on a stable set of probabilities in the first example, then apply our limiting results to the *g*-expectations.

From the perspective of probability theory, coherent risk measures can be regarded as appropriate nonlinear expectations, with a connection to numerous papers use non-additive probabilities or nonlinear expectations (for example Choquet expectation, upper expectation, *G*-expectation) to describe and interpret uncertainty in mathematical economics, statistics and finance, and develop limit theorems under different frameworks. Related literature includes Walley and Fine [4], Marinacci [5], Epstein and Schneider [6], Maccheroni and Marinacci [7], Cooman and Miranda [8], Chen *et al.* [9], Chen [10], Peng [11] and the references therein.

This paper is organized as follows. Section 2 introduces some important preliminary definitions and results for time-consistent dynamic coherent risk measures. Section 3 establishes the laws of large numbers for risk capacities and time-consistent dynamic coherent risk measures. In Section 4, we give examples where asymptotic behavior of the risk of returns in financial assets using a time-consistent dynamic coherent risk measure induced by a class of *g*-expectations is investigated. Section 5 gives a summary of these results.

2. The Model and Some Preliminary Results

Let $\mathbb{N}=\{0,1,2,\cdots\}$, we consider a filtered probability space $\left(\Omega,\mathcal{F},\{\mathcal{F}_i\}_{i\in\mathbb{N}},P\right)$ with $\mathcal{F}_0=\{\varnothing,\Omega\}$, where $\mathcal{F}_i$ models the information available at date *i*. Let $L^\infty:=L^\infty(\Omega,\mathcal{F},P)$ be the space of bounded random variables defined on $(\Omega,\mathcal{F},P)$, and $L_i^\infty:=L^\infty(\Omega,\mathcal{F}_i,P)$ defined similarly for each $i\in\mathbb{N}$. For each $p\ge 1$, let $L^p:=L^p(\Omega,\mathcal{F},P)=\{\xi:\xi$ is an $\mathcal{F}$-measurable random variable with $E_P\left[|\xi|^p\right]<\infty\}$, and $L_i^p:=L^p(\Omega,\mathcal{F}_i,P)$ defined similarly for each $i\in\mathbb{N}$. $\{X_i\}_{i=1}^\infty$ is a sequence of random variables with $X_i\in L_i^\infty$, where X_i describes the market return of portfolios at date *i* or a random net payoff to be delivered to an agent at that date. In this paper, all inequalities and equalities applied to random variables are meant to hold *P*-a.s..

We first introduce some concepts, which extends the original definition of coherent risk measure to the dynamic setting.

Definition 2.1 *A* ***dynamic risk measure*** *on* $\left(\Omega,\mathcal{F},\{\mathcal{F}_i\}_{i\in\mathbb{N}},P\right)$ *is a family* $(\rho_i)_{i\in\mathbb{N}}$ *of maps, where for each* $i\in\mathbb{N}$, *the map* $\rho_i:L^\infty\to L_i^\infty$ *satisfies the following properties*:

1) ***Monotonicity***: *For each* $X,Y\in L^\infty$, *if* $X\le Y$, *then* $\rho_i(X)\ge\rho_i(Y)$;

2) ***Translation invariance***: *For each* $X\in L^\infty$ *and* $Z\in L_i^\infty$,

$$\rho_i(X+Z)=\rho_i(X)-Z.$$

Definition 2.2 *A dynamic risk measure* $(\rho_i)_{i\in\mathbb{N}}$ *is called a dynamic* ***coherent*** *risk measure if for each* $i\in\mathbb{N}$, ρ_i *satisfies the following additional prop-*

erties:

3) ***Sub-additivity***: *For each* $X, Y \in L^{\infty}$, $\rho_i(X+Y) \le \rho_i(X) + \rho_i(Y)$;

4) ***Positive homogeneity***: *For each* $X \in L^{\infty}$ *and* $\lambda \ge 0$, $\rho_i(\lambda X) = \lambda \rho_i(X)$.

Definition 2.3 *A dynamic risk measure on* $(\Omega, \mathcal{F}, \{\mathcal{F}_i\}_{i\in\mathbb{N}}, P)$ *is called a* ***time-consistent*** *dynamic risk measure, if for each* $i \in \mathbb{N}$, ρ_i *satisfies the following property*:

5) ***Time consistency***: *For each* $X \in L^{\infty}$ *and* $j > i$, $\rho_i(X) = \rho_i(-\rho_j(X))$.

Remark 2.1 *Time consistency requires that judgements based on the risk measure are not contradictory over time. It means that the risk of each financial position can be indifferently evaluated directly or iteratively via some intermediate time. In some literature, Property* 5) *is also called recursiveness, and time consistency is defined as: for each* $X, Y \in L^{\infty}$,

$$\rho_{i+1}(X) = \rho_{i+1}(Y) \Rightarrow \rho_i(X) = \rho_i(Y), i \in \mathbb{N}.$$

When we consider a dynamic coherent risk measure, these two definitions are equivalent, see, e.g. [12].

Note that for a dynamic coherent risk measure $(\rho_i)_{i\in\mathbb{N}}$, when $i = 0$, $\rho_0 : L^{\infty} \to \mathbb{R}$ is a coherent risk measure. We can define a pair of risk capacities $(\mathbb{V}, v)$ on $(\Omega, \mathcal{F})$ based on a coherent risk measure ρ_0 as follows:

Definition 2.4 *Given a coherent risk measure* ρ_0, *for each* $A \in \mathcal{F}$, *let*

$$\mathbb{V}(A) := \rho_0(-I_A), v(A) := -\rho_0(I_A),$$

then $(\mathbb{V}, v)$ *is a pair of risk capacities generated by* ρ_0.

According to the properties of ρ_0, it is easy to check that this pair of risk capacities $(\mathbb{V}, v)$ satisfies the following properties 1)-2): for each $A, B \in \mathcal{F}$,

1) $\mathbb{V}(\varnothing) = v(\varnothing) = 0$, $\mathbb{V}(\Omega) = v(\Omega) = 1$;

2) If $A \subset B$, then $\mathbb{V}(A) \le \mathbb{V}(B), v(A) \le v(B)$;

3) $\mathbb{V}(A \cup B) \le \mathbb{V}(A) + \mathbb{V}(B)$;

4) $\mathbb{V}(A) + v(A^c) = 1$, where A^c is the complement set of *A*.

Example 2.1 *A very important leading case of the above concept of coherent risk measure is the following dynamic risk measures on a general set of probabilities. Given a measurable space* $(\Omega, \mathcal{F})$, *we denote the corresponding set of probability measures on this space by* $\Delta(\Omega, \mathcal{F})$. *If two probability measures define the same null sets, we say that these two measures are equivalent. Let* $(\Omega, \mathcal{F}, \{\mathcal{F}_i\}_{i\in\mathbb{N}}, P)$ *be the filtered probability space as we introduced in the beginning of this paper, we consider the following set* $\mathcal{P}$ *of probability measures*:

$$\mathcal{P} = \{Q \in \Delta(\Omega, \mathcal{F}) \mid Q \text{ is equivalent to } P\}.$$

For any random variable $X \in L^{\infty}$, *we define* $\rho_0^{\mathcal{P}}$ *on* $(\Omega, \mathcal{F}, \{\mathcal{F}_i\}_{i\in\mathbb{N}}, P)$ *as follows*,

$$\rho_0^{\mathcal{P}}(X) := \sup_{Q\in\mathcal{P}} E_Q[-X].$$

It is easy to check that $\rho_0^{\mathcal{P}}$ *is a coherent risk measure that can be applied to the case with model uncertainty on the probability measures that characterizes the* *stochastic behavior of the economic activity.*

Example 2.2 *Notice that the well-known conventional coherent risk measure Expected Shortfall—can be treated as a very special case of our general representation of risk measure. Given a probability space* $(\Omega, \mathcal{F}, P)$, *for any* $\tau \in (0,1)$, *let* $A_\tau = \{X \le F_X^{-1}(\tau)\}$, *where* F_X *is the CDF of X, we consider the following Choquet probability distortion*:

$$\mathbb{P}_\tau(\cdot) = \mathbb{P}(\cdot \mid A_\tau).$$

This type Choquet distorted probability measure reflects pessimism. The least favorable events receive increased weight and the most favorable events are discounted. In particular, the probabilities of the τ *least favorable outcomes are inflated and the* $1-\tau$ *proportions of most favorable outcomes are discounted entirely. The corresponding CDF is*

$$F_\tau(x) = \mathbb{P}_\tau(X \le x) = \frac{1}{\tau}\int_{(X \le x)\cap A_\tau} \mathrm{d}F_X(\cdot).$$

In particular, let the distortion function

$$v_\tau(t) = \min\left\{\frac{t}{\tau}, 1\right\},$$

then the distorted Choquet expected value is given by

$$\mathbb{E}_\tau(X) = \mathbb{E}_{\mathbb{P}_\tau}(X) = \int_{-\infty}^{\infty} x \mathrm{d} v_\tau(F_X(x)) = \frac{1}{\tau}\int_0^\tau F_X^{-1}(t)\mathrm{d}t = \mathbb{E}(X \mid A_\tau),$$

corresponding to the τ*-th expected shortfall (ES). We consider a class of probability measures generated based on this type distortions. For any given* $\alpha \in (0,1)$, *consider the following class of probability measures*

$$\mathcal{P} = \{\mathbb{P}_\tau : \tau \ge \alpha\},$$

i.e.

$$\mathcal{P} = \{\mathbb{P}_\tau : \tau \ge \alpha\} = \{\mathbb{P}(\cdot \mid A_\tau) : \tau \ge \alpha\}.$$

Then a coherent risk measure is given by

$$\begin{aligned}\rho(X) &= \sup_{\tau \ge \alpha} \mathbb{E}_{\mathbb{P}_\tau}(-X) = -\inf_{\tau \ge \alpha} \mathbb{E}_{\mathbb{P}_\tau}(X) \\ &= -\inf_{\tau \ge \alpha} \mathbb{E}\left(X \mid X \le F_X^{-1}(\tau)\right) \\ &= -\mathbb{E}\left(X \mid X \le F_X^{-1}(\alpha)\right),\end{aligned}$$

which corresponds to the α*-th expected shortfall* (*ES*).

The following two propositions are basic tools in the proof of our main results, and the first one extends the Proposition 1 in [12].

Proposition 2.1 *Let* $(\rho_i)_{i\in\mathbb{N}}$ *be a dynamic coherent risk measure, then for each* $i \in \mathbb{N}$, ρ_i *satisfies that for each* $Z \in L_i^\infty$ *and* $X \in L^\infty$,

$$\rho_i(ZX) = Z^+\rho_i(X) + Z^-\rho_i(-X),$$

where $Z^+ := ZI_{\{Z>0\}}$ *and* $Z^- := ZI_{\{Z\le 0\}}$.

Proof. When $i = 0$, this proposition can be deduced by the properties of ρ_0.

When $i \geq 1$, we prove this proposition in the following three steps.

Step 1: We show that for each $A \in \mathcal{F}_i$ and $X, Y \in L^\infty$,

$$\rho_i\left(XI_A + YI_{A^c}\right) = I_A \rho_i(X) + I_{A^c} \rho_i(Y).$$

By monotonicity and translation invariance of ρ_i,

$$I_A \rho_i(X) \leq I_A \rho_i\left(XI_A - \|X\|_{L^\infty} I_{A^c}\right) = I_A \rho_i(XI_A).$$

Similarly, $I_A \rho_i(X) \geq I_A \rho_i(XI_A)$, thus $I_A \rho_i(X) = I_A \rho_i(XI_A)$. Then

$$\begin{aligned}\rho_i\left(XI_A + YI_{A^c}\right) &= I_A \rho_i\left(XI_A + YI_{A^c}\right) + I_{A^c} \rho_i\left(XI_A + YI_{A^c}\right) \\ &= I_A \rho_i(X) + I_{A^c} \rho_i(Y).\end{aligned}$$

Step 2: Suppose that Z is a simple function, *i.e.* $Z = \sum_{j=1}^n x_j I_{A_j}$, where $x_j \in \mathbb{R}$ and $\{A_j\}_{j=1}^n$ is an $\mathcal{F}_i$-partition of Ω. We have

$$\begin{aligned}\rho_i(ZX) &= \rho_i\left(\sum_{j=1}^n x_j I_{A_j} X\right) = \sum_{j=1}^n I_{A_j} \rho_i\left(x_j X\right) \\ &= \sum_{j=1}^n I_{A_j}\left[x_j^+ \rho_i(X) + x_j^- \rho_i(-X)\right] \\ &= Z^+ \rho_i(X) + Z^- \rho_i(-X).\end{aligned}$$

Step 3: Let $Z \in L_i^\infty$, then there exists a sequence $\{Z_n\}_{n\geq1}$ of $\mathcal{F}_i$-measurable simple functions such that Z_n uniformly converge to Z. Then we have

$$\begin{aligned}\left|\rho_i(ZX) - \rho_i(Z_n X)\right| &\leq \rho_i\left(-|Z - Z_n||X|\right) \\ &\leq \rho_i\left(-|Z - Z_n|\|X\|_{L^\infty}\right) \\ &= \|X\|_{L^\infty} |Z - Z_n| \\ &\to 0.\end{aligned}$$

That is,

$$\begin{aligned}\rho_i(ZX) &= \lim_{n\to\infty} \rho_i(Z_n X) \\ &= \lim_{n\to\infty}\left[Z_n^+ \rho_i(X) + Z_n^- \rho_i(-X)\right] \\ &= Z^+ \rho_i(X) + Z^- \rho_i(-X).\end{aligned}$$

Proposition 2.2 *Let* $(\rho_i)_{i\in\mathbb{N}}$ *be a dynamic coherent risk measure, then for each* $i \in \mathbb{N}$, ρ_i *satisfies that for each* $X, Y \in L^\infty$,

$$\left|\rho_i(X) - \rho_i(Y)\right| \leq \rho_i\left(-|X - Y|\right).$$

Proof. For each $i \in \mathbb{N}$, by sub-additivity of ρ_i,

$$-\rho_i(Y - X) \leq \rho_i(X) - \rho_i(Y) \leq \rho_i(X - Y),$$

then

$$\left|\rho_i(X) - \rho_i(Y)\right| \leq \max\left\{\left|\rho_i(X - Y)\right|, \left|\rho_i(Y - X)\right|\right\}.$$

Since $\rho_i(X - Y) + \rho_i(Y - X) \geq 0$, by monotonicity of ρ_i,

$$\rho_i(X - Y) \leq \rho_i\left(-|X - Y|\right),$$

$$\rho_i(X - Y) \geq -\rho_i(Y - X) \geq -\rho_i\left(-|X - Y|\right),$$

thus, $\left|\rho_i\left(X-Y\right)\right| \le \rho_i\left(-\left|X-Y\right|\right)$. Similarly, we have $\left|\rho_i\left(Y-X\right)\right| \le \rho_i\left(-\left|X-Y\right|\right)$. Therefore,

$$\left|\rho_i\left(X\right)-\rho_i\left(Y\right)\right| \le \rho_i\left(-\left|X-Y\right|\right).$$

For a sequence of random variables $\{X_i\}_{i=1}^{\infty}$, we now give the definition of risk independence with respect to a dynamic risk measure $\left(\rho_i\right)_{i\in\mathbb{N}}$ on $\left(\Omega,\mathcal{F},\{\mathcal{F}_i\}_{i\in\mathbb{N}},P\right)$.

Definition 2.5 *Let* $\{X_i\}_{i=1}^{\infty}$ *be a sequence of random variables with* $X_i \in L_i^{\infty}$. *For a dynamic risk measure* $\left(\rho_i\right)_{i\in\mathbb{N}}$, X_{i+1} *is said to be risk independent of* $\left(X_1,\cdots,X_i\right)$, *if*

$$\rho_i\left(X_{i+1}\right)=\rho_0\left(X_{i+1}\right), \rho_i\left(-X_{i+1}\right)=\rho_0\left(-X_{i+1}\right).$$

$\{X_i\}_{i=1}^{\infty}$ *is said to be a sequence of risk independent random variables, if* X_{i+1} *is risk independent of* $\left(X_1,\cdots,X_i\right)$ *for each* $i\ge 1$.

Remark 2.2 *If* $\{X_i\}_{i=1}^{\infty}$ *is a sequence of risk independent random variables for a time-consistent dynamic risk measure* $\left(\rho_i\right)_{i\in\mathbb{N}}$, *we can easily obtain that for each* $0<j<i$,

$$\rho_j\left(X_i\right)=\rho_0\left(X_i\right), \rho_j\left(-X_i\right)=\rho_0\left(-X_i\right).$$

3. The Laws of Large Numbers

Let $\left(\rho_i\right)_{i\in\mathbb{N}}$ be a time-consistent dynamic coherent risk measure on $\left(\Omega,\mathcal{F},\{\mathcal{F}_i\}_{i\in\mathbb{N}},P\right)$, we consider a sequence $\{X_i\}_{i=1}^{\infty}$ of risk independent random variables with $X_i \in L_i^{\infty}$. In this section, we develop the asymptotic behavior of dynamic coherent risk measures based on the sample average of $\{X_i\}_{i=1}^{n}$.

Assume that $\sup_{i\ge 1}\rho_0\left(-\left|X_i\right|\right)<\infty$, and there exist two constants $\overline{\mu},\underline{\mu}\in\mathbb{R}$ such that for each $i\ge 1$,

$$\rho_0\left(X_i\right)\equiv-\underline{\mu}, \rho_0\left(-X_i\right)\equiv\overline{\mu},$$

then, by risk independence of $\{X_i\}_{i=1}^{\infty}$, we have, for each $i\ge 1$,

$$\rho_i\left(X_{i+1}\right)=-\underline{\mu}, \rho_i\left(-X_{i+1}\right)=\overline{\mu}.$$

In addition, we further assume that, for any given $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\rho_0\left(-\left|X_i\right|I_{\{|X_i|>n\varepsilon\}}\right)=0. \tag{1}$$

For notational convenience, we denote $S_0=0$ and $S_m:=\sum_{i=1}^{m}X_i$, for $m\ge 1$.

Let $C_b\left(\mathbb{R}\right)$ be the collection of bounded and continuous functions, and $C_b^2\left(\mathbb{R}\right)$ be the subset of $C_b\left(\mathbb{R}\right)$ of twice differentiable functions with bounded derivatives of all orders. We first give some preliminary results that are important for the derivation of our main Theorem 3.1.

Lemma 3.1 *Let* $\left(\rho_i\right)_{i\in\mathbb{N}}$ *be a time-consistent dynamic coherent risk measure defined on* L^{∞}. *Assume that* $\{f_m\}_{m\ge 1}\in C_b^2\left(\mathbb{R}\right)$ *is a sequence of functions and there exists a constant* $c>0$ *such that*

$$\sup_{m\ge 1}\sup_{x\in\mathbb{R}}\left|f_m'\left(x\right)\right|\le c,\ \sup_{m\ge 1}\sup_{x\in\mathbb{R}}\left|f_m''\left(x\right)\right|\le c. \tag{2}$$

1) *If* f_m *are increasing, then*

$$\lim_{n\to\infty}\sum_{m=1}^{n}\left|\rho_0\left(f_m\left(\frac{S_m}{n}\right)\right)-\rho_0\left(f_m\left(\frac{S_{m-1}}{n}\right)+\frac{\mu}{n}f'_m\left(\frac{S_{m-1}}{n}\right)\right)\right|=0. \quad (3)$$

2) *If* f_m *are decreasing, then*

$$\lim_{n\to\infty}\sum_{m=1}^{n}\left|\rho_0\left(f_m\left(\frac{S_m}{n}\right)\right)-\rho_0\left(f_m\left(\frac{S_{m-1}}{n}\right)+\frac{\overline{\mu}}{n}f'_m\left(\frac{S_{m-1}}{n}\right)\right)\right|=0. \quad (4)$$

Proof. We only give the proof of (3), the proof of (4) is similar.

By time consistency of $(\rho_i)_{i\in\mathbb{N}}$ and Proposition 2.1, it is easy to show that

$$\begin{aligned}&\rho_0\left(f_m\left(\frac{S_{m-1}}{n}\right)+f'_m\left(\frac{S_{m-1}}{n}\right)\frac{X_m}{n}\right)\\&=\rho_0\left(f_m\left(\frac{S_{m-1}}{n}\right)-\frac{1}{n}\rho_{m-1}\left(f'_m\left(\frac{S_{m-1}}{n}\right)X_m\right)\right)\\&=\rho_0\left(f_m\left(\frac{S_{m-1}}{n}\right)+\frac{\mu}{n}f'_m\left(\frac{S_{m-1}}{n}\right)\right).\end{aligned}$$

So, it suffices to prove

$$\lim_{n\to\infty}\sum_{m=1}^{n}\left|\rho_0\left(f_m\left(\frac{S_m}{n}\right)\right)-\rho_0\left(f_m\left(\frac{S_{m-1}}{n}\right)+f'_m\left(\frac{S_{m-1}}{n}\right)\frac{X_m}{n}\right)\right|=0.$$

By the Taylor expansion of f_m and the assumption (2), we have for each $\varepsilon>0$, there exist $\delta>0$ (δ depends only on c and ε) such that for any $x,y\in\mathbb{R}$, and $m\ge 1$,

$$\left|f_m(x+y)-f_m(x)-f'_m(x)y\right|\le\varepsilon|y|I_{\{|y|\le\delta\}}+2c|y|I_{\{|y|>\delta\}}. \quad (5)$$

Set $x=\frac{S_{m-1}}{n}$ and $y=\frac{X_m}{n}$, by Proposition 2.2 and monotonicity of ρ_0, we obtain

$$\begin{aligned}&\sum_{m=1}^{n}\left|\rho_0\left(f_m\left(\frac{S_m}{n}\right)\right)-\rho_0\left(f_m\left(\frac{S_{m-1}}{n}\right)+f'_m\left(\frac{S_{m-1}}{n}\right)\frac{X_m}{n}\right)\right|\\&\le\sum_{m=1}^{n}\rho_0\left(-\left|f_m\left(\frac{S_m}{n}\right)-f_m\left(\frac{S_{m-1}}{n}\right)-f'_m\left(\frac{S_{m-1}}{n}\right)\frac{X_m}{n}\right|\right)\\&\le L_1(\varepsilon,n)+L_2(c,n),\end{aligned}$$

where $L_1(\varepsilon,n)$ and $L_2(c,n)$ are denoted by

$$L_1(\varepsilon,n):=\frac{\varepsilon}{n}\sum_{m=1}^{n}\rho_0\left(-|X_m|I_{\{|X_m|\le n\delta\}}\right),$$

$$L_2(c,n):=\frac{2c}{n}\sum_{m=1}^{n}\rho_0\left(-|X_m|I_{\{|X_m|>n\delta\}}\right).$$

Since $\sup_{i\ge1}\rho_0(-|X_i|)<\infty$, by the arbitrary of ε and condition (1), it's easy to prove that

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\left(L_1(\varepsilon,n)+L_2(c,n)\right)=0.$$

The proof is complete.

Lemma 3.2 *Let* $(\rho_i)_{i\in\mathbb{N}}$ *be a time-consistent dynamic coherent risk measure defined on* L^∞*, then*

1) *For each increasing function* $\varphi\in C_b^2(\mathbb{R})$,

$$\lim_{n\to\infty}\rho_0\left(\varphi\left(\frac{S_n}{n}\right)\right)=-\varphi(\underline{\mu}). \tag{6}$$

2) *For each decreasing function* $\varphi\in C_b^2(\mathbb{R})$,

$$\lim_{n\to\infty}\rho_0\left(\varphi\left(\frac{S_n}{n}\right)\right)=-\varphi(\overline{\mu}). \tag{7}$$

Proof. We only prove (6), the proof of (7) is similar. Note that

$$\begin{aligned}
&\rho_0\left(\varphi\left(\frac{S_n}{n}\right)\right)-\left(-\varphi(\underline{\mu})\right)\\
&=\rho_0\left(\varphi\left(\frac{S_n}{n}\right)\right)-\rho_0\left(\varphi(\underline{\mu})\right)\\
&=\rho_0\left(\varphi\left(\frac{S_n}{n}\right)\right)-\rho_0\left(\varphi\left(\frac{S_{n-1}}{n}+\frac{1}{n}\underline{\mu}\right)\right)\\
&\quad+\rho_0\left(\varphi\left(\frac{S_{n-1}}{n}+\frac{1}{n}\underline{\mu}\right)\right)-\rho_0\left(\varphi\left(\frac{S_{n-2}}{n}+\frac{2}{n}\underline{\mu}\right)\right)+\cdots\\
&\quad+\rho_0\left(\varphi\left(\frac{S_m}{n}+\frac{n-m}{n}\underline{\mu}\right)\right)-\rho_0\left(\varphi\left(\frac{S_{m-1}}{n}+\frac{n-m+1}{n}\underline{\mu}\right)\right)+\cdots\\
&\quad+\rho_0\left(\varphi\left(\frac{S_1}{n}+\frac{n-1}{n}\underline{\mu}\right)\right)-\rho_0\left(\varphi(\underline{\mu})\right)\\
&=\sum_{m=1}^{n}\left\{\rho_0\left(\varphi\left(\frac{S_m}{n}+\frac{n-m}{n}\underline{\mu}\right)\right)-\rho_0\left(\varphi\left(\frac{S_{m-1}}{n}+\frac{n-m+1}{n}\underline{\mu}\right)\right)\right\}.
\end{aligned}$$

Let

$$f_m(x):=\varphi\left(x+\frac{n-m+1}{n}\underline{\mu}\right),\quad m=1,2,\cdots,n+1,$$

then

$$\begin{aligned}
&\rho_0\left(\varphi\left(\frac{S_n}{n}\right)\right)-\left(-\varphi(\underline{\mu})\right)\\
&=\sum_{m=1}^{n}\left\{\rho_0\left(f_{m+1}\left(\frac{S_m}{n}\right)\right)-\rho_0\left(f_m\left(\frac{S_{m-1}}{n}\right)\right)\right\}\\
&=\sum_{m=1}^{n}\left\{\rho_0\left(f_{m+1}\left(\frac{S_m}{n}\right)\right)-\rho_0\left(f_{m+1}\left(\frac{S_{m-1}}{n}\right)+\frac{\underline{\mu}}{n}f'_{m+1}\left(\frac{S_{m-1}}{n}\right)\right)\right\}\\
&\quad+\sum_{m=1}^{n}\left\{\rho_0\left(f_{m+1}\left(\frac{S_{m-1}}{n}\right)+\frac{\underline{\mu}}{n}f'_{m+1}\left(\frac{S_{m-1}}{n}\right)\right)-\rho_0\left(f_m\left(\frac{S_{m-1}}{n}\right)\right)\right\}\\
&=:I_{1n}+I_{2n}.
\end{aligned}$$

Since $\varphi\in C_b^2(\mathbb{R})$, we have $f_m\in C_b^2(\mathbb{R})$ and $\sup_{x\in\mathbb{R}}\left|f'_m(x)\right|$ and $\sup_{x\in\mathbb{R}}\left|f''_m(x)\right|$ are both bounded uniformly for all *m*. By Lemma 3.1, we have,

$$\left|I_{1n}\right| \le \sum_{m=1}^{n}\left|\rho_0\left(f_{m+1}\left(\frac{S_m}{n}\right)\right)-\rho_0\left(f_{m+1}\left(\frac{S_{m-1}}{n}\right)+\frac{\mu}{n}f'_{m+1}\left(\frac{S_{m-1}}{n}\right)\right)\right| \to 0,\ \text{as } n\to\infty.$$

Furthermore, by Proposition 2.2,

$$\begin{aligned}\left|I_{2n}\right| &\le \sum_{m=1}^{n}\rho_0\left(-\left|f_{m+1}\left(\frac{S_{m-1}}{n}\right)+\frac{\mu}{n}f'_{m+1}\left(\frac{S_{m-1}}{n}\right)-f_m\left(\frac{S_{m-1}}{n}\right)\right|\right)\\ &\le \sum_{m=1}^{n}\sup_{x\in\mathbb{R}}\left|f_{m+1}(x)+\frac{\mu}{n}f'_{m+1}(x)-f_m(x)\right|\\ &= \sum_{m=1}^{n}\sup_{x\in\mathbb{R}}\left|f_{m+1}\left(x+\frac{\mu}{n}\right)-f_{m+1}(x)-\frac{\mu}{n}f'_{m+1}(x)\right|\\ &\to 0,\ \text{as } n\to\infty.\end{aligned}$$

Then, we complete the proof.

With the above preliminary results, we can now prove our main result.

Theorem 3.1 (Law of Large Numbers) *Let* $(\rho_i)_{i\in\mathbb{N}}$ *be a time-consistent dynamic coherent risk measure defined on* L^∞, *and* $\{X_i\}_{i=1}^{\infty}$ *be a sequence of risk independent random variables such that for each* $i\ge 1$, $X_i\in L_i^\infty$ *and satisfies* $\sup_{i\ge1}\rho_0(-|X_i|)<\infty$,

$$\rho_0(X_i)\equiv-\underline{\mu},\ \rho_0(-X_i)\equiv\overline{\mu},\quad \underline{\mu}\le\overline{\mu}\in\mathbb{R}.$$

Assume further that for each $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\rho_0\left(-|X_i|I_{\{|X_i|>n\varepsilon\}}\right)=0. \tag{8}$$

Then we have,

1) *For each* $\varepsilon>0$,

$$\lim_{n\to\infty}\mathbb{V}\left(\left\{\frac{S_n}{n}\ge\overline{\mu}+\varepsilon\right\}\cup\left\{\frac{S_n}{n}\le\underline{\mu}-\varepsilon\right\}\right)=0.$$

2) *For each* $\varepsilon>0$ *and* $h\in[\underline{\mu},\overline{\mu}]$,

$$\lim_{n\to\infty}\mathbb{V}\left(h-\varepsilon<\frac{S_n}{n}<h+\varepsilon\right)=1.$$

3) *For each* $\varphi\in C_b(\mathbb{R})$,

$$\lim_{n\to\infty}\rho_0\left(\varphi\left(\frac{S_n}{n}\right)\right)=-\inf_{\underline{\mu}\le x\le\overline{\mu}}\varphi(x).$$

Proof. 1) For each $\varepsilon>0$, let $\phi(x)\in C_b^2(\mathbb{R})$ be an increasing function with $\phi(x)=-1$ when $x\le\underline{\mu}-\varepsilon$, $\phi(x)=0$ when $x\ge\underline{\mu}-\frac{\varepsilon}{2}$, and $-1<\phi(x)<0$ when $\underline{\mu}-\varepsilon<x<\underline{\mu}-\frac{\varepsilon}{2}$, then $-I_{\{x\le\underline{\mu}-\varepsilon\}}\ge\phi(x)$ and $\phi(\underline{\mu})=0$, combine with Lemma 3.2, we have

$$0\le\mathbb{V}\left(\frac{S_n}{n}\le\underline{\mu}-\varepsilon\right)\le\rho_0\left(\phi\left(\frac{S_n}{n}\right)\right)\to-\phi(\underline{\mu})=0,\ \text{as } n\to\infty.$$

Similarly, we have $\lim_{n\to\infty}\mathbb{V}\left(\frac{S_n}{n}\geq\overline{\mu}+\varepsilon\right)=0$, then

$$
\begin{aligned}
0 &\leq \lim_{n\to\infty}\mathbb{V}\left(\left\{\frac{S_n}{n}\geq\overline{\mu}+\varepsilon\right\}\cup\left\{\frac{S_n}{n}\leq\underline{\mu}-\varepsilon\right\}\right)\\
&\leq \lim_{n\to\infty}\mathbb{V}\left(\frac{S_n}{n}\geq\overline{\mu}+\varepsilon\right)+\lim_{n\to\infty}\mathbb{V}\left(\frac{S_n}{n}\leq\underline{\mu}-\varepsilon\right)\\
&=0.
\end{aligned}
$$

2) For each $\varepsilon>0$ and $h\in\left[\underline{\mu},\overline{\mu}\right]$, by the monotonicity of $\mathbb{V}$, we only need to prove

$$\liminf_{n\to\infty}\mathbb{V}\left(h-\varepsilon<\frac{S_n}{n}<h+\varepsilon\right)\geq 1.$$

Construct a function $\psi(x)\in C_b^2(\mathbb{R})$ such that, $\psi(x)=0$ when $x\leq h-\varepsilon$ or $x\geq h+\varepsilon$, and $-1\leq\psi(x)\leq 0$ when $h-\varepsilon<x<h+\varepsilon$, and $\psi(h)=-1$. It's easy to check that $-I_{\{h-\varepsilon<x<h+\varepsilon\}}\leq\psi(x)$ and

$$\mathbb{V}\left(h-\varepsilon<\frac{S_n}{n}<h+\varepsilon\right)\geq\rho_0\left(\psi\left(\frac{S_n}{n}\right)\right).$$

Since $\psi\in C_b^2(\mathbb{R})$, there exist a constant $c>0$ such that $\sup_{x\in\mathbb{R}}|\psi'(x)|\leq c$, $\sup_{x\in\mathbb{R}}|\psi''(x)|\leq c$. By the Taylor expansion of ψ, for each $\varepsilon>0$, there exist $\delta>0$ (δ depends only on c and ε) such that for any $x,y\in\mathbb{R}$,

$$\left|\psi(x+y)-\psi(x)-\psi'(x)y\right|\leq\varepsilon|y|I_{\{|y|\leq\delta\}}+2c|y|I_{\{|y|>\delta\}}.$$

Let $T_m:=\frac{S_{m-1}}{n}+\frac{n-m}{n}h$. By time consistency and coherency of $(\rho_i)_{i\in\mathbb{N}}$, and using condition (8), we have,

$$
\begin{aligned}
&\rho_0\left(\psi\left(\frac{S_n}{n}\right)\right)-\rho_0(\psi(h))\\
&=\sum_{m=1}^{n}\left\{\rho_0\left(\psi\left(\frac{S_m}{n}+\frac{n-m}{n}h\right)\right)-\rho_0\left(\psi\left(\frac{S_{m-1}}{n}+\frac{n-m+1}{n}h\right)\right)\right\}\\
&=\sum_{m=1}^{n}\left\{\rho_0\left(\psi\left(T_m+\frac{X_m}{n}\right)\right)-\rho_0\left(\psi\left(T_m+\frac{h}{n}\right)\right)\right\}\\
&\geq\sum_{m=1}^{n}\left\{\rho_0\left(\psi(T_m)+\psi'(T_m)\frac{X_m}{n}+\frac{\varepsilon}{n}|X_m|I_{\{|X_m|\leq n\delta\}}+\frac{2c}{n}|X_m|I_{\{|X_m|>n\delta\}}\right)\right.\\
&\quad\left.-\rho_0\left(\psi(T_m)+\psi'(T_m)\frac{h}{n}-\frac{\varepsilon}{n}|h|I_{\{|h|\leq n\delta\}}-\frac{2c}{n}|h|I_{\{|h|>n\delta\}}\right)\right\}\\
&\geq\sum_{m=1}^{n}\left\{\rho_0\left(\psi(T_m)+(\psi'(T_m))^+\frac{\underline{\mu}}{n}-(\psi'(T_m))^-\frac{\overline{\mu}}{n}\right)-\rho_0\left(\psi(T_m)+\psi'(T_m)\frac{h}{n}\right)\right.\\
&\quad\left.-\rho_0\left(-\frac{\varepsilon}{n}|X_m|I_{\{|X_m|\leq n\delta\}}-\frac{2c}{n}|X_m|I_{\{|X_m|>n\delta\}}\right)-\frac{\varepsilon}{n}|h|I_{\{|h|\leq n\delta\}}-\frac{2c}{n}|h|I_{\{|h|>n\delta\}}\right\}\\
&\geq\sum_{m=1}^{n}\left\{-\frac{\varepsilon}{n}\rho_0(-|X_m|)-\frac{2c}{n}\rho_0\left(-|X_m|I_{\{|X_m|>n\delta\}}\right)-\frac{\varepsilon}{n}|h|-\frac{2c}{n}|h|I_{\{|h|>n\delta\}}\right\}\\
&\to 0,\ \text{as } n\to\infty,\varepsilon\to 0.
\end{aligned}
$$

Then we have

$$\liminf_{n\to\infty} \mathbb{V}\left(h-\varepsilon<\frac{S_n}{n}<h+\varepsilon\right)\geq \rho_0\left(\psi\left(h\right)\right)=1.$$

3) If $\varphi\in C_b\left(\mathbb{R}\right)$, then for any $\varepsilon>0$, there exists $\bar{\varphi}\in C_b^2\left(\mathbb{R}\right)$ such that

$$\sup_{x\in\mathbb{R}}\left|\varphi\left(x\right)-\bar{\varphi}\left(x\right)\right|<\varepsilon.$$

So we only need to prove the result for the case where $\varphi\in C_b^2\left(\mathbb{R}\right)$.

First, as a consequence of 1), for each $\varepsilon>0$,

$$\begin{aligned}\rho_0\left(\varphi\left(\frac{S_n}{n}\right)\right)&\leq \rho_0\left(\varphi\left(\frac{S_n}{n}\right)I_{\left\{\underline{\mu}-\varepsilon<\frac{S_n}{n}<\bar{\mu}+\varepsilon\right\}}\right)+\rho_0\left(\varphi\left(\frac{S_n}{n}\right)I_{\left\{\frac{S_n}{n}\leq\underline{\mu}-\varepsilon\right\}\cup\left\{\frac{S_n}{n}\geq\bar{\mu}+\varepsilon\right\}}\right)\\&\leq -\inf_{\underline{\mu}-\varepsilon\leq x\leq\bar{\mu}+\varepsilon}\varphi\left(x\right)+\left\|\varphi\right\|\mathbb{V}\left(\left\{\frac{S_n}{n}\geq\bar{\mu}+\varepsilon\right\}\cup\left\{\frac{S_n}{n}\leq\underline{\mu}-\varepsilon\right\}\right)\\&\to -\inf_{\underline{\mu}-\varepsilon\leq x\leq\bar{\mu}+\varepsilon}\varphi\left(x\right),\end{aligned}$$

where $\left\|\varphi\right\|=\sup_{x\in R}\left|\varphi\left(x\right)\right|$. With the arbitrariness of $\varepsilon>0$ and the continuity of φ, we have

$$\limsup_{n\to\infty}\rho_0\left(\varphi\left(\frac{S_n}{n}\right)\right)\leq -\inf_{\underline{\mu}\leq x\leq\bar{\mu}}\varphi\left(x\right). \tag{9}$$

On the other hand, let x^* be the point in $\left[\underline{\mu},\bar{\mu}\right]$, such that $\varphi\left(x^*\right)=\inf_{\underline{\mu}\leq x\leq\bar{\mu}}\varphi\left(x\right)$. By the Taylor expansion of φ, for each $n\geq1$, there exists a random variable θ_n valued in $[0,1]$ such that

$$\varphi\left(\frac{S_n}{n}\right)-\varphi\left(x^*\right)=\varphi'\left(x^*+\theta_n\left(\frac{S_n}{n}-x^*\right)\right)\left(\frac{S_n}{n}-x^*\right).$$

Thanks to 2), for each ε, we have

$$\begin{aligned}&\rho_0\left(\varphi\left(\frac{S_n}{n}\right)\right)-\left(-\inf_{\underline{\mu}\leq x\leq\bar{\mu}}\varphi\left(x\right)\right)\\&=\rho_0\left(\varphi\left(\frac{S_n}{n}\right)-\varphi\left(x^*\right)\right)\\&=\rho_0\left(\left(\varphi\left(\frac{S_n}{n}\right)-\varphi\left(x^*\right)\right)I_{\left\{\left|\frac{S_n}{n}-x^*\right|<\varepsilon\right\}}+\left(\varphi\left(\frac{S_n}{n}\right)-\varphi\left(x^*\right)\right)I_{\left\{\left|\frac{S_n}{n}-x^*\right|\geq\varepsilon\right\}}\right)\\&=\rho_0\left(\varphi'\left(x^*+\theta_n\left(\frac{S_n}{n}-x^*\right)\right)\left(\frac{S_n}{n}-x^*\right)I_{\left\{\left|\frac{S_n}{n}-x^*\right|<\varepsilon\right\}}+\left(\varphi\left(\frac{S_n}{n}\right)-\varphi(x^*)\right)I_{\left\{\left|\frac{S_n}{n}-x^*\right|\geq\varepsilon\right\}}\right)\\&\geq\rho_0\left(\left\|\varphi'\right\|\varepsilon+2\left\|\varphi\right\|I_{\left\{\left|\frac{S_n}{n}-x^*\right|\geq\varepsilon\right\}}\right)\\&=-\left\|\varphi'\right\|\varepsilon+2\left\|\varphi\right\|\left[\mathbb{V}\left(x^*-\varepsilon<\frac{S_n}{n}<x^*+\varepsilon\right)-1\right]\\&\to-\left\|\varphi'\right\|\varepsilon.\end{aligned}$$

By arbitrariness of $\varepsilon > 0$,

$$\liminf_{n\to\infty} \rho_0\left(\varphi\left(\frac{S_n}{n}\right)\right) \geq -\inf_{\underline{\mu}\leq x\leq\bar{\mu}} \varphi(x).$$

Combine the above result with (9), we obtain

$$\lim_{n\to\infty} \rho_0\left(\varphi\left(\frac{S_n}{n}\right)\right) = -\inf_{\underline{\mu}\leq x\leq\bar{\mu}} \varphi(x).$$

4. Examples

The asymptotic results in Section 3 can be applied to a wide range of settings. In this section, we give two specific examples of time-consistent dynamic coherent risk measures to illustrate the potential of our previous results. Our first example provides some additional investigations to the dynamic risk measure giving in Example 2.1 of Section 2. The second example considers the *g*-expectations.

4.1. Dynamic Coherent Risk Measures in the Presence of Model Uncertainty

We first consider the risk measure given in Example 2.1 in Section 2 in the presence of model uncertainty. As mentioned in Section 2, we can verify that $\rho_0^{\mathcal{P}}$ is a coherent risk measure. Next, we show that, under appropriate conditions, there exits a unique time-consistent dynamic coherent risk measure $\left(\rho_i^{\mathcal{P}}\right)_{i\in\mathbb{N}}$ on $\left(\Omega,\mathcal{F},\{\mathcal{F}_i\}_{i\in\mathbb{N}},P\right)$.

Artzner *et al.* [13] and Delbaen [14] prove that a dynamic coherent risk measure defined from a set of probability measures is time-consistent if and only if this set satisfies the stability condition. Riedel [15] calls this condition also "consistency", Roorda *et al.* [16] uses "product property", and Epstein and Schneider [6] name it "rectangular" in decision theoretic framework.

Bion-Nadal ([17]) gives the following definition of stable set $\mathcal{P}$, which is weaker than that of [14].

Definition 4.1 *Let* $\mathcal{P}$ *be a set of probability measures on* $(\Omega,\mathcal{F})$ *all equivalent to P.* $\mathcal{P}$ *is stable if the following conditions hold.*

1) *Stability by composition*: *For each* $0\leq m\leq n$ *and* $R,S\in\mathcal{P}$, *there is a probability measure* $Q\in\mathcal{P}$ *such that for each random variable* $X\in L^\infty$,

$$E_Q\left[X\mid\mathcal{F}_m\right] = E_R\left[E_S\left[X\mid\mathcal{F}_n\right]\mid\mathcal{F}_m\right];$$

2) *Stability by bifurcation: For each* $n\in\mathbb{N}$, $R,S\in\mathcal{P}$ *and* $A\in\mathcal{F}_n$, *there is a probability measure* $Q\in\mathcal{P}$ *such that for each random variable* $X\in L^\infty$,

$$E_Q\left[X\mid\mathcal{F}_n\right] = I_A E_R\left[X\mid\mathcal{F}_n\right] + I_{A^c} E_S\left[X\mid\mathcal{F}_n\right].$$

Proposition 4.1 *If* $\mathcal{P}$ *is stable, there exists a family* $\left(\rho_i^{\mathcal{P}}\right)_{i\geq1}$ *such that* $\left(\rho_i^{\mathcal{P}}\right)_{i\in\mathbb{N}}$ *is a time-consistent dynamic coherent risk measure defined on* L^∞. *Moreover, if* $\mathcal{P}$ *is weakly compact in* L^1 *norm, the existence is unique under P-equivalence.*

Proof. 1) **Existence:** For any random variable $X\in L^\infty$, define a family $\left(\rho_i^{\mathcal{P}}\right)_{i\geq1}$

as follows,

$$\rho_i^{\mathcal{P}}(X) := \operatorname*{ess\,sup}_{P\in\mathcal{P}} E_P[-X \mid \mathcal{F}_i], \forall i \geq 1.$$

It is easy to check that $(\rho_i^{\mathcal{P}})_{i\in\mathbb{N}}$ satisfies the conditions 1)-4) given in Definition 2.1. So we only need to prove that

$$\rho_j^{\mathcal{P}}(X) = \rho_j^{\mathcal{P}}(-\rho_i^{\mathcal{P}}(X)), 0 \leq j < i. \tag{10}$$

Fix some $X \in L^\infty$ and $i \geq 1$. For any $P_1, P_2 \in \mathcal{P}$, let

$$A = \{E_{P_1}[-X \mid \mathcal{F}_i] > E_{P_2}[-X \mid \mathcal{F}_i]\} \in \mathcal{F}_i.$$

From stability by bifurcation, we can construct a probability measure $P_3 \in \mathcal{P}$ such that

$$E_{P_3}[-X \mid \mathcal{F}_i] = I_A E_{P_1}[-X \mid \mathcal{F}_i] + I_{A^c} E_{P_2}[-X \mid \mathcal{F}_i],$$

that is $\{E_Q[-X \mid \mathcal{F}_i] \mid Q \in \mathcal{P}\}$ is a lattice upward directed. According to Appendix A.4 in [18],

$$\rho_i^{\mathcal{P}}(X) = \lim_{n\to\infty} E_{P_n}[-X \mid \mathcal{F}_i],$$

where $P_n \in \mathcal{P}$ and $\{E_{P_n}[-X \mid \mathcal{F}_i]\}$ is an increasing sequence of random variables.

Similarly, we can prove that for any given $0 \leq j < i$, there exists a sequence $\{Q_m\}_{m\geq 1}$, $Q_m \in \mathcal{P}$, such that $\rho_j^{\mathcal{P}}(-\rho_i^{\mathcal{P}}(X))$ is the increasing limit of $E_{Q_m}[\rho_i^{\mathcal{P}}(X) \mid \mathcal{F}_j]$.

Then by monotonic convergence theorem,

$$\begin{aligned}
\rho_j^{\mathcal{P}}(-\rho_i^{\mathcal{P}}(X)) &= \lim_{m\to\infty} E_{Q_m}\left[\lim_{n\to\infty} E_{P_n}[-X \mid \mathcal{F}_i] \mid \mathcal{F}_j\right] \\
&= \lim_{m,n\to\infty} E_{Q_m}\left[E_{P_n}[-X \mid \mathcal{F}_i] \mid \mathcal{F}_j\right] \\
&\leq \operatorname*{ess\,sup}_{Q\in\mathcal{P}} E_Q[-X \mid \mathcal{F}_j] \\
&= \rho_j^{\mathcal{P}}(X),
\end{aligned}$$

the inequality dues to the stability by composition of $\mathcal{P}$. Conversely,

$$\begin{aligned}
\rho_j^{\mathcal{P}}(X) &= \operatorname*{ess\,sup}_{Q\in\mathcal{P}} E_Q[-X \mid \mathcal{F}_j] \\
&= \operatorname*{ess\,sup}_{Q\in\mathcal{P}} E_Q\left[E_Q[-X \mid \mathcal{F}_i] \mid \mathcal{F}_j\right] \\
&\leq \operatorname*{ess\,sup}_{Q\in\mathcal{P}} E_Q\left[\operatorname*{ess\,sup}_{R\in\mathcal{P}} E_R[-X \mid \mathcal{F}_i] \mid \mathcal{F}_j\right] \\
&= \rho_j^{\mathcal{P}}(-\rho_i^{\mathcal{P}}(X)),
\end{aligned}$$

Thus,

$$\rho_j^{\mathcal{P}}(X) = \rho_j^{\mathcal{P}}(-\rho_i^{\mathcal{P}}(X)), 0 \leq j < i.$$

2) **Uniqueness:** Suppose there exist two families $\{\rho_0^{\mathcal{P}}, (\rho_i^{1,\mathcal{P}})_{i\geq 1}\}$ and $\{\rho_0^{\mathcal{P}}, (\rho_i^{2,\mathcal{P}})_{i\geq 1}\}$ are time-consistent dynamic coherent risk measures.

Fix some $i \geq 1$. By Proposition 2.1 and Equation (10), for each $A \in \mathcal{F}_i$ and $X \in L^\infty$, we have $I_A \in L_i^\infty$ and

$$\rho_0^{\mathcal{P}}\left(-I_A \rho_i^{1,\mathcal{P}}(X)\right) = \rho_0^{\mathcal{P}}\left(-I_A \rho_i^{2,\mathcal{P}}(X)\right).$$

In particular, let $A_n = \left\{\rho_i^{1,\mathcal{P}}(X) > \rho_i^{2,\mathcal{P}}(X) + \frac{1}{n}\right\}$, $n \geq 1$, then A_n is $\mathcal{F}_i$-measurable, and we have

$$\rho_0^{\mathcal{P}}\left(-I_{A_n} \rho_i^{1,\mathcal{P}}(X)\right) = \rho_0^{\mathcal{P}}\left(-I_{A_n} \rho_i^{2,\mathcal{P}}(X)\right).$$

Since

$$\begin{aligned}\rho_0^{\mathcal{P}}\left(-I_{A_n} \rho_i^{1,\mathcal{P}}(X)\right) &\geq \rho_0^{\mathcal{P}}\left(-I_{A_n}\left(\rho_i^{2,\mathcal{P}}(X) + \frac{1}{n}\right)\right) \\ &\geq \rho_0^{\mathcal{P}}\left(-I_{A_n} \rho_i^{2,\mathcal{P}}(X)\right) - \rho_0^{\mathcal{P}}\left(I_{A_n} \frac{1}{n}\right),\end{aligned}$$

we have

$$\rho_0^{\mathcal{P}}\left(I_{A_n}\right) = 0.$$

Furthermore, there exist a sequence $\{P_m\}_{m\geq 1}$ such that

$$\lim_{m\to\infty} E_{P_m}\left[-I_{A_n}\right] = \rho_0^{\mathcal{P}}\left(I_{A_n}\right). \tag{11}$$

If $\mathcal{P}$ is weakly compact in L^1 norm, then for this sequence $\{P_m\}_{m\geq 1}$, there exist a subsequence $\{P_{m_k}\}_{k\geq 1}$ with Radon-Nikodym derivatives $\frac{\mathrm{d}P_{m_k}}{\mathrm{d}P}$ such that

$$E_P\left[\left|\frac{\mathrm{d}P_{m_k}}{\mathrm{d}P} - \frac{\mathrm{d}Q}{\mathrm{d}P}\right|\right] \to 0, \text{ as } k \to \infty,$$

for some probability measure $Q \in \mathcal{P}$. Then

$$E_{P_{m_k}}\left[-I_{A_n}\right] = E_P\left[-I_{A_n} \frac{\mathrm{d}P_{m_k}}{\mathrm{d}P}\right] \to E_P\left[-I_{A_n} \frac{\mathrm{d}Q}{\mathrm{d}P}\right] = E_Q\left[-I_{A_n}\right], \text{ as } k \to \infty.$$

This combines with (11) to obtain

$$E_Q\left[-I_{A_n}\right] = \rho_0^{\mathcal{P}}\left(I_{A_n}\right) = 0.$$

Therefore, $Q(A_n) = 0$ and

$$Q\left(\rho_i^{1,\mathcal{P}}(X) > \rho_i^{2,\mathcal{P}}(X)\right) = \lim_{n\to\infty} Q(A_n) = 0.$$

By the equivalence of $\mathcal{P}$, we have $\rho_i^{1,\mathcal{P}}(X) \leq \rho_i^{2,\mathcal{P}}(X)$. Similarly, we can obtain $\rho_i^{1,\mathcal{P}}(X) \geq \rho_i^{2,\mathcal{P}}(X)$. That is $\left(\rho_i^{\mathcal{P}}(X)\right)_{i\geq 1}$ is unique under *P*-equivalence.

Theorem 4.1 *Let* $\mathcal{P}$ *be stable and* $\{X_i\}_{i=1}^\infty$ *be a sequence of random variables such that for each* $i \geq 1$, $X_i \in L_i^\infty$ *and satisfies* $\sup_{i\geq 1} \sup_{P\in\mathcal{P}} E_P\left[|X_i|\right] < \infty$, *and* $\operatorname{ess\,sup}_{Q\in\mathcal{P}} E_Q\left[X_i \mid \mathcal{F}_{i-1}\right] = \overline{\mu}$,

$\operatorname{ess\,inf}_{Q\in\mathcal{P}} E_Q\left[X_i \mid \mathcal{F}_{i-1}\right]=\underline{\mu}$, $\underline{\mu}\le\overline{\mu}\in\mathbb{R}$. *Assume that for each* $\epsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sup_{P\in\mathcal{P}}E_P\left[\left|X_i\right|I_{\{|X_i|>n\epsilon\}}\right]=0.$$

Then we have, for each $\varphi\in C_b(\mathbb{R})$,

$$\lim_{n\to\infty}\rho_0^{\mathcal{P}}\left(\varphi\left(\frac{S_n}{n}\right)\right)=-\inf_{\underline{\mu}\le x\le\overline{\mu}}\varphi(x).$$

4.2. Dynamic Risk Measures Based on g-Expectations

g-expectations are introduced by Peng [19] via a class of nonlinear Backward Stochastic Differential Equations (BSDEs). They are classical examples of time-consistent dynamic risk measures. Rosazza Gianin [20] provides some sufficient conditions for a *g*-expectation to be a dynamic coherent risk measure

Given an integer $n\in\mathbb{N}$ and let $(W_t)_{0\le t\le n}$ be an 1-dimensional standard Brownian motion defined on a completed probability space $(\Omega,\mathcal{F},P)$. Suppose $\{\mathcal{F}_t\}_{0\le t\le n}$ is the natural filtration generated by $(W_t)_{0\le t\le n}$, *i.e.* $\mathcal{F}_t=\sigma\{W_s,0\le s\le t\}\vee\mathcal{N}$, where $\mathcal{N}$ is the collection of *P*-null subsets. We also assume $\mathcal{F}_n=\mathcal{F}$. Consider the following BSDE

$$Y_t=X+\int_t^n\kappa\left|\sigma_s\right|\mathrm{d}s-\int_t^n\sigma_s\mathrm{d}W_s, \tag{12}$$

where $X\in L^2$ and $\kappa\in\mathbb{R}^+$ (refer to [21] κ-Ignorance). Pardoux and Peng [22] show that BSDE (12) has a unique adapted solution $(Y_t,\sigma_t)_{t\ge0}\in L^2(0,n,\mathbb{R})\times L^2(0,n,\mathbb{R})$, where $L^2(0,n,\mathbb{R}):=\{X_t: X_t$ is a $\mathbb{R}$-valued and $\{\mathcal{F}_t\}$-adapted process with $E_P\left[\int_0^n\left|X_s\right|^2\mathrm{d}s\right]<\infty\}$.

The *g*-expectation of *X* is defined by

$$\mathcal{E}_g[X]:=Y_0,$$

and for each $t\in[0,n]$, the conditional *g*-expectation of *X* under $\mathcal{F}_t$ is defined by

$$\mathcal{E}_g[X\mid\mathcal{F}_t]:=Y_t.$$

For some $n\in\mathbb{N}$ large enough, consider the finite set of dates $\mathbb{N}^n=\{i,i=0,\cdots,n\}$ when the risks of market values are assessed, and a discrete filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}^n}$ models the information available at date *i*. For each random variable $X\in L^2$, we define a family $\left(\rho_i^g\right)_{i\in\mathbb{N}^n}$ on $\left(\Omega,\mathcal{F},(\mathcal{F}_i)_{i\in\mathbb{N}^n},P\right)$ as follows,

$$\rho_0^g(X):=\mathcal{E}_g[-X],$$

$$\rho_i^g(X):=\mathcal{E}_g[-X\mid\mathcal{F}_i],1\le i\le n.$$

By Peng [19] and Chen *et al.* [23], we know $\left(\rho_i^g\right)_{i\in\mathbb{N}^n}$ is a time-consistent dynamic coherent risk measure defined on L^2.

Lemma 4.1 *For each* $i\in\mathbb{N}^n$, ρ_i^g *satisfies that for each* $Z\in L_i^\infty$ *and* $X\in L^2$,

$$\rho_i^g(ZX) = Z^+\rho_i^g(X) + Z^-\rho_i^g(-X),$$

where $Z^+ := ZI_{\{Z>0\}}$ *and* $Z^- := ZI_{\{Z\le 0\}}$.

Proof. Fix some $X \in L^2$. First, consider any simple function $Z := \sum_{j=1}^m x_j I_{A_j} \in L_i^\infty$, where $x_j \in \mathbb{R}$ and $\{A_j\}_{j=1}^m$ is an $\mathcal{F}_i$-partition of Ω. By Lemma 2 in Chen and Peng [24] and positive homogeneity of ρ_i^g, we have

$$\begin{aligned}\rho_i^g(ZX) &= \sum_{j=1}^m I_{A_j}\rho_i^g(ZX)\\ &= \sum_{j=1}^m I_{A_j}\rho_i^g(x_jX)\\ &= \sum_{j=1}^m \left(x_j^+ I_{A_j}\rho_i^g(X) + x_j^- I_{A_j}\rho_i^g(-X)\right)\\ &= Z^+\rho_i^g(X) + Z^-\rho_i^g(-X).\end{aligned} \tag{13}$$

Next, consider any random variable $Z \in L_i^\infty$, there exists an increasing sequence $\{Z_m\}_{m\ge1}$ of $\mathcal{F}_i$-measurable simple functions such that Z_m uniformly converge to Z and $|Z_m| \le \|Z\|_\infty$. Since $Z_mX \to ZX$, *P-a.s.* and $|Z_mX| \le \|Z\|_\infty |X|$, by dominated convergence theorem, we have

$$E_P\left[|Z_mX - ZX|^2\right] \to 0, \text{ as } m \to \infty.$$

From standard estimates of BSDEs [25], we have

$$E_P\left[\left|\rho_i^g(ZX) - \rho_i^g(Z_mX)\right|^2\right] \le CE_P\left[|Z_mX - ZX|^2\right] \to 0, \text{ as } m \to \infty,$$

where C is a constant independent of *m*. Then there exists a subsequence $\{Z_{m_k}\}_{m_k\ge1}$ of $\{Z_m\}_{m\ge1}$ satisfying

$$\lim_{k\to\infty}\rho_i^g(Z_{m_k}X) = \rho_i^g(ZX).$$

According to Equation (13), we have

$$\lim_{k\to\infty}\rho_i^g(Z_{m_k}X) = \lim_{k\to\infty}\left(Z_{m_k}^+\rho_i^g(X) + Z_{m_k}^-\rho_i^g(-X)\right) = Z^+\rho_i^g(X) + Z^-\rho_i^g(-X).$$

Thus,

$$\rho_i^g(ZX) = Z^+\rho_i^g(X) + Z^-\rho_i^g(-X).$$

Remark 4.1 *By this lemma, the condition* $X_i \in L_i^\infty$, $i \ge 1$, *in Theorem 3.1, can be extended to* $X_i \in L_i^2$.

The next example illustrates how our law of large numbers works in evaluating the risk of a financial asset by $(\rho_i^g)_{i\in\mathbb{N}^n}$.

Theorem 4.2 *Let* $(\rho_i^g)_{i\in\mathbb{N}^n}$ *be defined as above and* S_t *be the value process of some financial asset having the following geometric Brownian motion characterization,*

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t,\ S_0 = s_0 \in \mathbb{R}^+, \tag{14}$$

where μ *and* $\sigma > 0$. *Then we have for each* $\varphi \in C_b(\mathbb{R})$,

$$\lim_{n\to\infty}\rho_0^g\left(\varphi\left(\frac{\ln S_n}{n}\right)\right) = -\inf_{x\in A}\varphi(x),$$

where $A=\left[\mu-\frac{\sigma^2}{2}-\sigma\kappa,\mu-\frac{\sigma^2}{2}+\sigma\kappa\right]$.

Proof. By solving the SDE (14),

$$S_t=s_0\exp\left[\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t\right],$$

and $\ln S_t=\ln s_0+\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t$ satisfies normal distribution $N\left(\ln s_0+\left(\mu-\frac{\sigma^2}{2}\right)t,\sigma^2 t\right)$. Let $X_i:=\ln S_i-\ln S_{i-1}\in L_i^2$, $1\le i\le n$. We can verify the conditions in Theorem 3.1 as follows.

1) $\sup_{i\ge1}\rho_0^g\left(-\left|X_i\right|\right)<\infty$, and for each $0\le i\le n$, $\rho_i^g\left(X_{i+1}\right)=-\mu+\frac{\sigma^2}{2}+\sigma\kappa$, $\rho_i^g\left(-X_{i+1}\right)=\mu-\frac{\sigma^2}{2}+\sigma\kappa$.

For each random variable $X\in L^2$, by Chen and Epstein [21], we have

$$\rho_0^g\left(X\right)=\sup_{Q^v\in\mathcal{P}}E_{Q^v}\left[-X\right],$$

where

$$\mathcal{P}:=\left\{Q^v:E\left[\frac{\mathrm{d}Q^v}{\mathrm{d}P}\middle|\mathcal{F}_t\right]:=\mathrm{e}^{-\frac{1}{2}\int_0^t v_s^2\mathrm{d}s+\int_0^t v_s\mathrm{d}W_s},\left|v\right|\le\kappa\right\}.$$

For each process v_t bounded by κ, *i.e.* $\left|v\right|\le\kappa$, let $\tilde{W}_t:=W_t-\int_0^t v_s\mathrm{d}s$, $t\ge0$, then by Girsanov's theorem $\tilde{W}_t$ is a Q^v-Brownian motion under $Q^v\in\mathcal{P}$. Therefore,

$$\begin{aligned}\sup_{i\ge1}\rho_0^g\left(-\left|X_i\right|\right)&=\sup_{i\ge1}\sup_{Q^v\in\mathcal{P}}E_{Q^v}\left[\left|\mu-\frac{\sigma^2}{2}+\sigma\left(W_i-W_{i-1}\right)\right|\right]\\&\le\left|\mu-\frac{\sigma^2}{2}\right|+\sigma\sup_{i\ge1}\sup_{Q^v\in\mathcal{P}}E_{Q^v}\left[\left|W_i-W_{i-1}\right|\right]\\&=\left|\mu-\frac{\sigma^2}{2}\right|+\sigma\sup_{i\ge1}\sup_{Q^v\in\mathcal{P}}E_{Q^v}\left[\left|\tilde{W}_i-\tilde{W}_{i-1}+\int_{i-1}^i v_s\mathrm{d}s\right|\right]\\&\le\left|\mu-\frac{\sigma^2}{2}\right|+\sigma+\sigma\kappa.\end{aligned}$$

Since X_{i+1} is independent of $\mathcal{F}_i$, we have

$$\rho_i^g\left(X_{i+1}\right)=\rho_0^g\left(X_{i+1}\right)=\sup_{Q^v\in\mathcal{P}}E_{Q^v}\left[-X_{i+1}\right].$$

Since

$$\begin{aligned}\sup_{Q^v\in\mathcal{P}}E_{Q^v}\left[-X_{i+1}\right]&=-\mu+\frac{\sigma^2}{2}+\sigma\sup_{Q^v\in\mathcal{P}}E_{Q^v}\left[-\left(\tilde{W}_{i+1}-\tilde{W}_i+\int_i^{i+1}v_s\mathrm{d}s\right)\right]\\&\le-\mu+\frac{\sigma^2}{2}+\sigma\kappa,\end{aligned}$$

meanwhile, for $v_t \equiv -\kappa$, $t \geq 0$, we have $E_{Q^v}\left[-X_{i+1}\right] = -\mu + \frac{\sigma^2}{2} + \sigma\kappa$. Then

$$\rho_i^g\left(X_{i+1}\right) = -\mu + \frac{\sigma^2}{2} + \sigma\kappa.$$

Similarly, consider $-X_{i+1}$, we have

$$\rho_i^g\left(-X_{i+1}\right) = \mu - \frac{\sigma^2}{2} + \sigma\kappa.$$

2) For each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \rho_0^g\left(-\left|X_i\right| I_{\{|X_i|>n\varepsilon\}}\right) = 0.$$

Fix some $\varepsilon > 0$, we also have, for n large enough,

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\rho_0^g\left(-\left|X_i\right| I_{\{|X_i|>n\varepsilon\}}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\sup_{Q^v\in\mathcal{P}} E_{Q^v}\left[\left|X_i\right| I_{|X_i|>n\varepsilon}\right] \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\sup_{Q^v\in\mathcal{P}} E_{Q^v}\left[\left|\mu - \frac{\sigma^2}{2} + \sigma\left(\tilde{W}_i - \tilde{W}_{i-1} + \int_{i-1}^{i} v_s \mathrm{d}s\right)\right| I_{\left\{\left|\tilde{W}_i - \tilde{W}_{i-1} + \int_{i-1}^{i} v_s \mathrm{d}s\right| > \frac{2n\varepsilon - \left|2\mu - \sigma^2\right|}{2\sigma}\right\}}\right] \\
&\leq \frac{\sigma}{n}\sum_{i=1}^{n}\sup_{Q^v\in\mathcal{P}} E_{Q^v}\left[\left|\tilde{W}_i - \tilde{W}_{i-1}\right| I_{\left\{\left|\tilde{W}_i - \tilde{W}_{i-1}\right| > \frac{2n\varepsilon - \left|2\mu - \sigma^2\right| - 2\sigma\kappa}{2\sigma}\right\}}\right] \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\sup_{Q^v\in\mathcal{P}} E_{Q^v}\left[\left(\mu + \frac{\sigma^2}{2} + \sigma\int_{i-1}^{i}\left|v_s\right| \mathrm{d}s\right) I_{\left\{\left|\tilde{W}_i - \tilde{W}_{i-1}\right| > \frac{2n\varepsilon - \left|2\mu - \sigma^2\right| - 2\sigma\kappa}{2\sigma}\right\}}\right] \\
&\leq \frac{2\sigma^2}{n\left(2n\varepsilon - \left|2\mu - \sigma^2\right| - 2\sigma\kappa\right)^2}\sum_{i=1}^{n}\sup_{Q^v\in\mathcal{P}} E_{Q^v}\left[\left|\tilde{W}_i - \tilde{W}_{i-1}\right|^2\right] \\
&\quad + \frac{2\sigma^2\left(2\mu + \sigma^2 + 2\sigma\kappa\right)}{n\left(2n\varepsilon - \left|2\mu - \sigma^2\right| - 2\sigma\kappa\right)^2}\sum_{i=1}^{n}\sup_{Q^v\in\mathcal{P}} E_{Q^v}\left[\left|\tilde{W}_i - \tilde{W}_{i-1}\right|^2\right] \\
&\to 0, \text{ as } n \to \infty.
\end{aligned}$$

Therefore, the sequence $\{X_i\}$ satisfies all conditions assumed in Theorem 3.1, and we can obtain that for each $\varphi \in C_b(\mathbb{R})$,

$$\lim_{n\to\infty} \rho_0^g\left(\varphi\left(\frac{\ln S_n}{n}\right)\right) = -\inf_{x\in A}\varphi(x),$$

where $A = \left[\mu - \frac{\sigma^2}{2} - \sigma\kappa, \mu - \frac{\sigma^2}{2} + \sigma\kappa\right]$.

5. Summary

One of the most important properties of dynamic coherent risk measures is the time consistency, which guarantees judgements of agents based on the risk measure which are not contradictory over time. By this feature, we study the asymptotic behavior of general dynamic coherent risk measures regardless of the specific representations and propose three types of law of large numbers (LLN) for the average values of portfolios, which describe the limit behavior of portfolio risks, and provide a new theoretical basis for the numerical calculation of portfolio risks.

Theorem 3.1 shows that the limit of average returns of portfolios over time will generally be multivalued, with their limit point confined in a deterministic set.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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