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About Degeneration of Landau's Levels

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Abstract

Through the non separable solution of the eigenvalue problem associated to the problem of a charged particle in a flat box and a constant transversal magnetic field, with Landau and symmetric gauges, it is found that the Landau's levels are numerably degenerated in both cases. A mathematical proposition is proven to carry out this statement.

Keywords

Landau's Gauge, Symmetric Gauge, Quantum Hall Effect, Degeneration

1. Introduction

The quatum Hall effect has had a great deal of physical and experimental importance since its discovery [1] [2] [3] [4], and one of the basic elements to understand this effect is the Landau's levels (eigenvalues of the eigenvalue problem), which has shown being correct even if the eigenfunctions are not totally right since the eigenvalue problem is not separable in all of its variables [5]. These eigenfunctions have been also found in different form [6], but both of them result to be equivalents [7]. However, a correct non separable solution of the eigenvalue problem has already been given on references [5] [8], where it is shown that for the eigenvalue problem with the Hamiltonian

$$\hat{H} = \frac{\left(\mathbf{p} - q\mathbf{A}/c\right)^2}{2m}, \tag{1}$$

with the Landau's gauge $\mathbf{A} = \left(-By, 0, 0\right)$ (B is constant) and with the symmetric gauge $\mathbf{A} = B\left(y, -x, 0\right)/2$ (inverse magnetic field), the Landau's levels are gotten

$$E_n = \hbar\omega_c\left(n + 1/2\right), \quad \omega_c = qB/mc, \tag{2}$$

the magnetic flux Φ is quantized

$$\frac{\Phi}{\Phi_0} \in \mathcal{Z}, \quad \Phi_0 = 2\pi\hbar c/2q, \tag{3}$$

and the eigenfunctions for Landau's gauge (ignoring the *z*-variable) are given by

$$\Phi_n^L(x,y) = \sqrt{\frac{\beta}{L_y}} e^{-i\beta^2 xy} \psi_n(\beta x), \quad \beta = \sqrt{m\omega_c/\hbar}, \tag{4}$$

where L_y represents the length of the box in the *y*-direction, and ψ_n is the harmonic oscillator solutions. For the symmetric gauge the eigenfunctions are

$$\Phi_n^S(x,y) = A_n e^{-\alpha\left(x^2+y^2\right)-\lambda(x-iy)} \left(2\alpha(x+iy)+\lambda\right)^n, \quad \alpha = qB/4\hbar c, \tag{5}$$

where λ is a complex constant, and $A_n = e^{-|\lambda|^2/4\alpha} \Big/ \sqrt{\pi n!(2\alpha)^{n-1}}$ is a normalized constant. In addition, for the Landau's gauge case one has $\left[\hat{p}_x, \hat{H}\right] = 0$, and for the symmetric gauge case one has $\left[\hat{L}_z, \hat{H}\right] = 0$. These facts allow to have the following additionals generated functions

$$\hat{p}_x \Phi_n^L = m\omega_c(ix-y)\Phi_n^L - i\hbar\beta\sqrt{2n}\Phi_{n-1}^L \tag{6}$$

and

$$\hat{L}_z \Phi_n^S = \hbar\lambda z^* \Phi_n^S + \hbar\sqrt{2\alpha n}\, z\Phi_{n-1}^S, \quad z = x+iy, \tag{7}$$

which are also eigenfunctions of the Hamiltonian

$$\hat{H}\left(\hat{p}_x \Phi_n^L\right) = E_n\left(\hat{p}_x \Phi_n^L\right) \tag{8}$$

and

$$\hat{H}\left(\hat{L}_z \Phi_n^S\right) = E_n\left(\hat{L}_z \Phi_n^S\right). \tag{9}$$

from these relations, it was thought that Landau's levels were doubly degenerated. However, we will see that this result is deeper than it was first thought since it allows to have numerably degeneration for the Landau's levels. The reason for example (6) is also an eigenfunction, as shown in (8), is the following: from the expression $\left[\hat{p}_x, \hat{H}\right] = 0$, one has

$$0 = \left[\hat{p}_x, \hat{H}\right]\Phi_n^L = \hat{p}_x\left(\hat{H}\Phi_b^L\right) - \hat{H}\left(\hat{p}_x\Phi_n^L\right) = E_n\left(\hat{p}_x\Phi_n^L\right) - \hat{H}\left(\Phi_n^L\right), \tag{10}$$

and the result (8) follows.

2. Analysis of the Degeneration

Let us make first some mathematical statements that will help to understand the situation. Let $\mathcal{E}$ be our Hilbert space, normally the set of quadratic integrable function in some set Ω contained in some dimensional space $L^2(\Omega) = \int_\Omega |f|^2 \, d\mu$, and let $\mathcal{L}(\mathcal{E})$ the set of linear operators acting in the space $\mathcal{E}$. Thus, one has the following proposition.

Prop. 1.- Let $A, H \in \mathcal{L}(\mathcal{E})$ be linear operators such that $[A,H] = 0$, and let $\{E_n, \phi_n\}_{n\in\mathcal{Z}}$ be the solutions of the eigenvalue problem $H\phi = E\phi$. If $A\phi_n$ is not proportional to ϕ_n, then $A^j\phi_n, j \in \mathcal{Z}^+$ is an eigenfunction of H with the same eigenvalue E_n (Therefore, the spectrum is numerably degenerated).

Proof: The fact $\left[A,H\right]=0$ implies that $\left[A^j,H\right]=0$ for $j\in\mathcal{Z}^+$, where A^j means *j*-applications of the operator A ($A\circ A\circ\cdots\circ A$). Therefore one has that $H\left(A^j\phi_n\right)=E_n\left(A^j\phi_n\right)$. Since $A\phi_n$ is not proportional to ϕ_n, it represents a new function, and by induction $A^j\phi_n$ represents a new function for $j\in\mathcal{Z}^+$. So, defining $f_n^0=\phi_n$ and $f_n^j=A^j\phi_n$, one has a set functions $\left\{f_n^j\right\}_{n,j\in\mathcal{Z}^+}$ which are eigenfunctions with the same eigenvalue

$$Hf_n^j=E_nf_n^j,\quad n,j\in\mathcal{Z}^+\ \bullet \tag{11}$$

In addition to the above proposition, one has the following: If $H\in\mathcal{L}(\mathcal{E})$ is an Hermitian operator, that is $\left\langle Hf,g\right\rangle=\left\langle f,Hg\right\rangle$ with the inner product $\left\langle f,g\right\rangle=\int_\Omega f^*g\mathrm{d}\mu$, one has the known proposition

Prop. 2.- Let $H\in\mathcal{L}(\mathcal{E})$ be an Hermitian operator, and let $\left\{E_n,f_n^j\right\}$ the set of solutions of the eigenvalue problem $H\phi=E\phi$, where the spectrum is degenerated (this degeneration is represente by the index "*j*"). Then, the functions $\left\{f_n^j\right\}$ are orthogonal with respect the index "*n*", but the orthogonality is undetermined with respect the index "*j*".

Proof: The relation $\left\langle Hf_{n_1}^{j_1},f_{n_2}^{j_2}\right\rangle=\left\langle f_{n_1}^{j_1},Hf_{n_2}^{j_2}\right\rangle$ implies that $\left(E_{n_2}-E_{n_1}\right)\left\langle f_{n_1}^{j_1},f_{n_2}^{j_2}\right\rangle=0$. Then, for $n_1\neq n_2$ one has necessarily that $\left\langle f_{n_1}^{j_1},f_{n_2}^{j_2}\right\rangle=0$ (orthogonality, independently of j_1 and j_2), but if $n_1=n_2=n$ the expression $\left\langle f_{n_1}^{j_1},f_{n_2}^{j_2}\right\rangle$ is undetermined $\bullet$

Of course, given a non orthogonal set of functions $\left\{f_n^j\right\}$, one can construct an orthogonal set $\left\{\tilde f_n^j\right\}$ through the Gram-Schmidt process [9]. Now, the results presented in (4), (5), (6), (7), (8) and (9) state exactly the conditions for the application of the Prop. 1 above. Therefore, the Landau's levels are numerably degenerated in both cases with the Landau and symmetric gauges. The states associated to each Landau's level are

$$\left\{f_{n,j}^L\right\}_{j\in\mathcal{Z}^+},\quad f_{n,0}^L(x,y)=\Phi_n^L(x,y)\quad\text{and}\quad f_{n,j}^L=\hat p_x^j\Phi_n^L \tag{12}$$

and

$$\left\{f_{n,j}^S\right\}_{j\in Z^+},\quad f_{n,0}^S(x,y)=\Phi_n^S(x,y)\quad\text{and}\quad f_{n,j}^S=\hat L_z^j\Phi_n^S. \tag{13}$$

It is easy to see, for example, from (4) and (6) that $\left\langle f_n^0\middle|f_n^1\right\rangle\nsim\delta_{0,1}$. Therefore, the set defined by (12) is non orthogonal, and the same happens with the set (13). Of course, the general solution of the Schödinger's equation ($i\hbar\partial\Psi/\partial t=\hat H\Psi$) for this problem should be written for the Landau's gauge (ignoring the *z*-variable) as

$$\Psi^L(x,y,t)=\sum_{n,j=0}^{\infty}C_{n,j}^Lf_{n,j}^L(x,y)\mathrm{e}^{-iE_nt/\hbar} \tag{14}$$

and for the symmetric gauge as

$$\Psi^S(x,y,t)=\sum_{n,j=0}^{\infty}C_{n,j}^Sf_{n,j}^S(x,y)\mathrm{e}^{-iE_nt/\hbar}, \tag{15}$$

being $C_{n,j}^L$ and $C_{n,j}^S$ constant, and E_n is the Landau's levels (2).

Now, from the result (11) and the expression (12) it is not difficult to see that one has the following relation

$$f_n^{j+1} = m\omega_c\left(\hbar j f_n^{j-1} + (ix - y) f_n^j\right) - i\sqrt{2nm\omega_c\hbar}\, f_{n-1}^j, \quad j \geq 0, \tag{16}$$

and

$$\hat{H} f_n^{j+1} = E_n f_n^{j+1} + m\omega_c\left(\hbar j f_n^{j-1} + ix f_n^j + \frac{i}{m\omega_c}\hat{p}_y f_n^j\right). \tag{17}$$

Thus, from the result (11), one must have that

$$\hbar j f_n^{j-1} + ix f_n^j + \frac{i}{m\omega_c}\hat{p}_y f_n^j = 0, \tag{18}$$

which it is not difficult to check it directly.

Similarly, from (5) and (7), one can get

$$f_n^{j+1} = \hbar\lambda z^* \sum_{m=0}^{j} c_m^j f_n^m + \hbar\sqrt{2\alpha n} z \sum_{m=0}^{j} d_m^j f_{n-1}^m, \tag{19}$$

where one has defined the constants c_m^j and d_m^j as $c_m^j = \binom{j}{m}(-\hbar)^{j-m}$ and $d_m^j = \binom{j}{m}\hbar^{j-m}$, being $\binom{j}{m} = j!/m!(j-m)!$ the binomial coefficient. In addition, one has the following action

$$\begin{aligned}\hat{\mathbf{H}} f_n^{j+1} = E_n f_n^{j+1} - \frac{2\hbar^2}{m}\sum_{m=0}^{j}\Big(& c_m^j\hbar\lambda\partial_z f_n^m + c_m^j\alpha\hbar\lambda z^* f_n^m + d_m^j\hbar\sqrt{2\alpha n}\partial_{z^*} f_{n-1}^m \\ &+ d_m^j\alpha\hbar z\sqrt{2\alpha n} f_{n-1}^m\Big).\end{aligned} \tag{20}$$

Then, using (11), it follows that

$$\sum_{m=0}^{j}\left(c_m^j\hbar\lambda\partial_z f_n^m + c_m^j\alpha\hbar\lambda z^* f_n^m + d_m^j\hbar\sqrt{2\alpha n}\partial_{z^*} f_{n-1}^m + d_m^j\alpha\hbar z\sqrt{2\alpha n} f_{n-1}^m\right) = 0, \tag{21}$$

which it is also not difficult to verify directly.

3. Conclusion

Due to previous results (6), (7), (8) and (9), obtained in [8], and the Prop. 1, we must conclude that the Landau's levels are numerably degenerated. This degeneration may have important consequences in the quantum dynamics of the quantum Hall Effect and topological insulators.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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