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Resurrecting the Prospect of Supplementary Variables with the Principle of Local Realism

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Abstract: I produce a coherent mathematical formulation of the supplementary variables structure for Aspect's experimental test of Bell's inequality as devised by Clauser, Horne, Shimony, and Holt, a formalization which has been widely considered to be impossible. Contrary to Aspect's understanding, it is made clear that a supplementary variable formulation can represent any tendered probability distribution whatsoever. This includes both the QM distribution and the "naive distribution", which he had suggested as a foil. It has long been known that quantum theory does not support a complete distribution for the components of the thought experiment that underlies the inequality. However, further than that, here I identify precisely the bounding polytope of distributions that do cohere with both its explicit premises *and* with the prospect of supplementary variables. In this context, it is found once again that every distribution within this polytope respects the conditions of Bell's inequality, and that the famous evaluation of the gedankenexpectation defying it as $2\sqrt{2}$ is mistaken. The argument is relevant to all subsequent embellishments of experimental methodology post Aspect, designed to block seven declared possible loopholes. The probabilistic prognostications of quantum theory are not denied, nor are the experimental observations. However, their inferential implications have been misrepresented.

Keywords: no go theorems; Aspect/Bell inequality; CHSH form; completion of quantum theory; fundamental theorem of probability; linear programming

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1. Introduction

Deliberations of the past half century regarding Bell's inequality have led to a widely accepted proposition that a non-contradictory formulation of Einstein's supplementary variables interpretation of quantum theory is impossible. As presented in his memorial review of the situation some twenty years later, Alain Aspect [1] claimed to have established that quantum theory conflicts with any supplementary parameter theory since it violates the inequality. Further, he identified the culprit in the violation as the principle of local realism.

A recent publication [2] has challenged this conclusion, exposing a mathematical error of neglect in Aspect's analysis. He had identified the expectation of a crucial quantity arising in the form of the problem devised by by Clauser, Horne, Shimony, and Holt (CHSH) [3] as $E(s) = 2\sqrt{2}$, which is outside of the bounding interval of $[-2, +2]$ required by Bell. It has now been shown, rather, that the deliberations of quantum theory yield only an interval bound on the expectation $E(s)$ as $(1.1213, 2]$, completely within the required interval. This conclusion was reached without reference to the proposition of supplementary variables at all, relying only on the structure of the gedankenexperiment involved under the presumption of local realism.

My attention in this present article focuses very specifically on a mathematical formalization of the supplementary variables proposition, yet reaches an identical conclusion. Without reference to the extensive contestable literature on related foundational matters, I shall turn directly to the mathematical issues at hand, proceeding with a constructive analysis, which I now foretell.

Once I briefly review the setup of Aspect's gedankenexperiment on a single pair of photons to begin Section 2, I shall embellish the structure he proposed to complete its characterization and analysis in terms of the proposition of supplementary variables. We shall find that this characterization can support the assertion of *any* probability distribution whatsoever regarding the four paired outcomes of the experiment, QM probabilities included. This understanding contrasts with that of Aspect, who thought that quantum theoretical probabilities are not congruent with hidden variables, while another probability distribution he termed the "naive" distribution *is* congruent. Further, the assessment will make evident that every coherent distribution supports an expected value of the CHSH quantity lying within its required bounds. None of the cohering distributions for all four polarization gedankenproducts admits the quantum theoretic probability specifications for real experiments as its marginal distributions. Of course, only one of the experiments can actually be conducted, and the quantum probabilities are surely appropriate to it. This result concurs with the early recognition by Fine, though I believe he overstated his case in some respects, which will be recognized. In Section 3, I formulate a linear programming structure that identifies bounds on the quantum expectation $E(s)$ in keeping with the supplementary variables proposal. It yields only an interval for the expectation as $(1.1213, 2]$ rather than the mistaken unique evaluation of $2\sqrt{2}$ that is commonly subscribed. The presentation will conclude in Section 4 with a discussion of pertinent issues.

2. Supplementary Variables' Explanation of Quantum Behavior

In formalizing the mathematical structure of the supplementary variables proposal, I shall use the notation of Aspect's setup, and expand upon it. An experiment is conducted on a pair of photons traveling in opposite directions along an axis, $\mathbf{z}$, from a common source. The direction of one photon traveling toward observation station A on the left is opposite to the direction the paired photon travels toward station B on the right: $\mathbf{z}_A = -\mathbf{z}_B$. At the end of their respective journeys, the photon paths are recorded by detectors identifying whether each of them passes through or is deflected by a polarizer, angled in the (x, y) plane perpendicular to the incident photon, either in direction $\mathbf{a}$ or $\mathbf{a}'$ at station A and in direction $\mathbf{b}$ or $\mathbf{b}'$ at station B. This setup yields a specific relative angle between the polarizer directions at A and B, as viewed in a common coordinate system. Using notation that parentheses around a pair of directions denotes the angle between them, the determination of the angles $(\mathbf{a}, \mathbf{z}_A)$ and $(\mathbf{b}, \mathbf{z}_B)$ implies the relative angle between the polarization directions at the two stations as $(\mathbf{a}, \mathbf{b})$.

Suppose we begin exactly as does Aspect (with his Equation (17) from the 2002 presentation) by defining the following quantity as a function of λ. It is computed from results of a gedankenexperiment on the polarization of a single pair of photons in four different relative angle settings:

$$\begin{aligned} s(\lambda) &= A(\lambda,\mathbf{a})\,B(\lambda,\mathbf{b}) \;-\; A(\lambda,\mathbf{a})\,B(\lambda,\mathbf{b}') \;+\; A(\lambda,\mathbf{a}')\,B(\lambda,\mathbf{b}) \;+\; A(\lambda,\mathbf{a}')\,B(\lambda,\mathbf{b}')\,, \\ &= A(\lambda,\mathbf{a})\;[B(\lambda,\mathbf{b}) - B(\lambda,\mathbf{b}')] \;+\; A(\lambda,\mathbf{a}')\;[B(\lambda,\mathbf{b}) + B(\lambda,\mathbf{b}')], \quad \text{for } \lambda \in \Lambda \end{aligned} \tag{1}$$

where components of the vector λ are numerical indicators of "hidden variables", discussed below. The space of all possible values of such hidden variables is denoted by Λ, of whatever dimension might be appropriate. The component functions $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$ have the form

$$\begin{aligned} A(\lambda,\mathbf{a}) &= +1\;(\lambda \in \Lambda_{\mathbf{a}+}) \;-1\;(\lambda \in \Lambda_{\mathbf{a}-}), \quad \text{and} \\ B(\lambda,\mathbf{b}) &= +1\;(\lambda \in \Lambda_{\mathbf{b}+}) \;-1\;(\lambda \in \Lambda_{\mathbf{b}-}) \end{aligned} \tag{2}$$

where the subspace pairs $(\Lambda_{\mathbf{a}+}, \Lambda_{\mathbf{a}-})$ and $(\Lambda_{\mathbf{b}+}, \Lambda_{\mathbf{b}-})$ provide distinct partitions of Λ.

Here and throughout this work, the use of parentheses surrounding a mathematical statement that may be either true or false denotes the indicator value of 1 if the statement is true, and 0 if it is false. This convention allows that the values of $A(\lambda,\mathbf{a})$ and $B(\lambda,\mathbf{b})$ might each equal only either -1 or $+1$.

The two partitions of Λ are understood to be distinct, so their Cartesian product identifies a 4-constituent partition of

$$\Lambda = \Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+} \cup \Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}-} \cup \Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}+} \cup \Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}-} \tag{3}$$

There would be good reason to modify the notation of $A(\lambda, \mathbf{a})$ to appear as $A_{\mathbf{a}}(\lambda)$, since the variability of the observation value of A is proposed to depend on the supplementary variables λ, while the observation procedure is merely conducted in the condition of the polarizer direction selected as $\mathbf{a}$ rather than $\mathbf{a}'$. However, we shall continue with Aspect's notation to provide precise correspondence with his argument.

The factorizations appearing in the second line of Equation (1) rely on the principle of local realism. That is, when the observation of $A(\lambda, \mathbf{a})$ is made in consort with the observation $B(\lambda, \mathbf{b})$, as required to evaluate the first summand of $s(\lambda)$, whatever value A takes in this experiment is understood to be the same numerical value it would take on if it were to be observed in consort with $B(\lambda, \mathbf{b}')$, appearing in the second summand of $s(\lambda)$. For the supplementary variables pertinent to the determination of $A(\lambda, \mathbf{a})$ would be the same in both instantiations. Although the quantum probabilities for the observation values of A and B do depend explicitly on the relative angles $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{a}, \mathbf{b}')$ in two such experimental situations, it is this presumed principle of local realism that permits us to factor the individual observation value of $A(\lambda, \mathbf{a})$ out of these two terms. The same remark would pertain to the observation $A(\lambda, \mathbf{a}')$ in its two instances in (1).

The idea behind the formalization of the observation functions $A(\lambda, \mathbf{a})$ and $B(\lambda, \mathbf{b})$ in (2) is that the summary information we have explained regarding the polarizer directions is deemed insufficient to describe the situation of the quantum experiment completely. In our current state of technology, it is hard to imagine how we might ever be able to observe further relevant features of the experimental "happening" during the physical transaction when the polarizing activities are being recorded. However, it is proposed that if we knew enough of such details of the experimental operation, currently unknown, we would be able to know more precisely what the spin observations would be. That is what the hidden variable dimensions of the incompleteness argument for quantum theory are all about, providing motivation for continuing speculative research investigation among advocates.

Such a scenario would characterize the mechanics of quantum activity in a similar fashion to that of physical activity at the scale of everyday life. Einstein claimed that the structure of the situations at quantum and classical scales is actually the same. It is just that at the quantum scale, we cannot even specify what all the relevant conditioning variables are, much less their precise values during the conduct of any experiment. John Bell originally surmised the same thing, though he was bemused by his inequality and puzzled by its apparent violation. Despite its relegation in mainstream quantum literature, Ed Jaynes [4] was an eminent advocate of the proposition of supplementary variables as relevant to the characterization of quantum phenomena for reasons he explained well, and he supported their investigation. He has not been alone. By now there are a number of fronts on which research has been and is being engaged, though results are controversial and sometimes merely speculative. It is true that several once-promising results have been disconfirmed, and doubters may regard the project as grasping at straws. However, it is hard to imagine that specification of the relative angle between the polarizer directions at recording stations would exhaust all that could *possibly* be known about experiments on paired photons.

Rather than dismissal as impossible, it would seem that further investigation of this matter and similar matters regarding electron spin observations induced by angled Stern–Gerlach magnets ought to be welcomed. Moreover, such relegation is hardly merited by the mistaken defiance of Bell's inequality by quantum probabilities in a gedankenexperiment. Mention should be made of the recognized efforts of Gerard 't Hooft, which are described in [5], among those of several others. However, a serious literature review of the current status of all lines of research is beyond the scope of the present article. My aim here is merely to formalize a mathematical structure for proposals of supplementary variables that cannot be dismissed as impossible. The coherent bounds for the CHSH expectation $E(s)$ will emerge as a byproduct, agreeing with results appearing in [2], which did not rely on allusion to supplementary variables at all.

2.1. Formalities of the Supplementary Variables Partition

In Aspect's formulation, numerical summary measures of such supplementary variables are represented by the variable vector λ. For a polarizer set up in direction $\mathbf{a}$, for example, the function $A(\lambda, \mathbf{a})$ represents the detection recorded at A under the hidden conditions that would be measured as λ, if they were measured. The domain of *all possible valuations* of such λ vector variables is denoted by Λ. It can be separated into two exclusive and exhaustive pieces, denoted by $\Lambda_{\mathbf{a}+}$ and $\Lambda_{\mathbf{a}-}$, these being the collections of such possibilities that would give rise to a measurement of $A(\lambda, \mathbf{a}) = +1$ and to $A(\lambda, \mathbf{a}) = -1$, respectively. A similar partition of the supplementary variables space Λ would correspond to the measurements at station B, specifying complementary components $\Lambda_{\mathbf{b}+}$ and $\Lambda_{\mathbf{b}-}$.

The quantity $s(\lambda)$ of Equation (1) pertains to instantiations of paired-photon products not only at the relative polarization angle $(\mathbf{a}, \mathbf{b})$, but at all four such angles in which either or both of the polarizer directions $\mathbf{a}$ and $\mathbf{b}$ are replaced by $\mathbf{a}'$ and $\mathbf{b}'$. Formalizing the space Λ of supplementary variables requires greater refinement to address the expectation of $s(\lambda)$. The form of component function pairs $A(\lambda, \mathbf{a})$ and $B(\lambda, \mathbf{b})$ pertains in exactly the same way to pairs in which either and/or both of the polarizer directions $\mathbf{a}$ and $\mathbf{b}$ are varied. In accounting for them all, we would have four such paired functions defined and observed, corresponding to relative polarizer angles $(\mathbf{a}, \mathbf{b})$, $(\mathbf{a}, \mathbf{b}')$, $(\mathbf{a}', \mathbf{b})$, and $(\mathbf{a}', \mathbf{b}')$. The cartesian product of *all four* such Λ partitions then yield a refined 16-constituent partition of Λ, viz.,

$$
\begin{aligned}
\Lambda \;=\; & \Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+}\Lambda_{\mathbf{a}'+}\Lambda_{\mathbf{b}'+} \cup \Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+}\Lambda_{\mathbf{a}'+}\Lambda_{\mathbf{b}'-} \cup \Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+}\Lambda_{\mathbf{a}'-}\Lambda_{\mathbf{b}'+} \cup \Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+}\Lambda_{\mathbf{a}'-}\Lambda_{\mathbf{b}'-} \\
& \cup \Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}+}\Lambda_{\mathbf{a}'+}\Lambda_{\mathbf{b}'+} \cup \Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}+}\Lambda_{\mathbf{a}'+}\Lambda_{\mathbf{b}'-} \cup \Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}+}\Lambda_{\mathbf{a}'-}\Lambda_{\mathbf{b}'+} \cup \Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}+}\Lambda_{\mathbf{a}'-}\Lambda_{\mathbf{b}'-} \\
& \cup \Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}-}\Lambda_{\mathbf{a}'+}\Lambda_{\mathbf{b}'+} \cup \Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}-}\Lambda_{\mathbf{a}'+}\Lambda_{\mathbf{b}'-} \cup \Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}-}\Lambda_{\mathbf{a}'-}\Lambda_{\mathbf{b}'+} \cup \Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}-}\Lambda_{\mathbf{a}'-}\Lambda_{\mathbf{b}'-} \\
& \cup \Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}-}\Lambda_{\mathbf{a}'+}\Lambda_{\mathbf{b}'+} \cup \Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}-}\Lambda_{\mathbf{a}'+}\Lambda_{\mathbf{b}'-} \cup \Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}-}\Lambda_{\mathbf{a}'-}\Lambda_{\mathbf{b}'+} \cup \Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}-}\Lambda_{\mathbf{a}'-}\Lambda_{\mathbf{b}'-}
\end{aligned} \tag{4}
$$

It is worth noticing here for future reference that the union of only the first four constituents of this partition equals the first constituent of the simpler partition displayed in (3). The unions of the next three rows would identify the other constituents of that partition. Moreover, the unions of other groups of carefully chosen components of (4) would provide us with the three other partitions of Λ relevant to the component experiments of the gedankenexperiment, as we shall find to be useful. Taking unions *down the columns* of the display would identify another easy partition for an example:

$$\Lambda \;=\; \Lambda_{\mathbf{a}'+}\Lambda_{\mathbf{b}'+} \cup \Lambda_{\mathbf{a}'+}\Lambda_{\mathbf{b}'-} \cup \Lambda_{\mathbf{a}'-}\Lambda_{\mathbf{b}'+} \cup \Lambda_{\mathbf{a}'-}\Lambda_{\mathbf{b}'-}\ .$$

We shall need to poke around to find the two others. For later reference in this endeavor, let us enumerate the partition components of Equation (4) as Λ_1 through Λ_{16}, numbering them sequentially across the rows as they appear there.

Considering all possible direction pairings for the polarizers at A and B (these being $\mathbf{a}$ or $\mathbf{a}'$, and $\mathbf{b}$ or $\mathbf{b}'$, respectively) along with the possible spin measurements of $+1$ or -1 at each end of any such pairing, the domain Λ is thus partitioned into the 16 constituents whose members are listed in the partition Equation (4). According to the imagination of the hidden variables proposition, the conceivably observable but hidden value of the λ vector would need be found to be within one of these sixteen constituents of its domain partition. Whichever one it happens to be, the value of $s(\lambda)$ would be determined. For examples, evaluating the summands of Bell's quantity according to the functions specified, we find

if $\lambda \in \Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}-}\Lambda_{\mathbf{a}'+}\Lambda_{\mathbf{b}'+}$, then $s(\lambda) = (-1)(-1) - (-1)(+1) + (+1)(-1) + (+1)(+1) = +2$, and
if $\lambda \in \Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}-}\Lambda_{\mathbf{a}'+}\Lambda_{\mathbf{b}'+}$, then $s(\lambda) = (+1)(-1) - (+1)(+1) + (+1)(-1) + (+1)(+1) = -2$, and
if $\lambda \in \Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}+}\Lambda_{\mathbf{a}'-}\Lambda_{\mathbf{b}'-}$, then $s(\lambda) = (-1)(+1) - (-1)(-1) + (-1)(+1) + (-1)(-1) = -2$.

Evaluating the value of $s(\lambda)$ for the λ values in every one of the constituents of the partition of Λ would show that the only possible values for s are -2 and $+2$, identifying the realm of gedankenobservation possibilities as $\mathcal{R}(s) = \{-2, +2\}$. It is this restriction which induces four symmetric functional relations among the four components of the linear combination specifying $s(\lambda)$. If any three of them sum to 3 or -1, then the fourth *must* equal -1 to meet the restriction limiting $s(\lambda)$ to equal only either -2 or $+2$. On the other hand, if the three sum to -3 or $+1$ then the fourth must equal $+1$. Suppose we call this function $G(\cdot,\cdot,\cdot)$, the G nominalizing it appropriately as a "gedankenfunction". As one example of the four, $A(\mathbf{a}')B(\mathbf{b}') = G[A(\mathbf{a})B(\mathbf{b}), A(\mathbf{a})B(\mathbf{b}'), A(\mathbf{a}')B(\mathbf{b})]$. The consequences of these functional relations will arise naturally in our consideration of the possibility space for the supplementary variables.

Now, it is clear that *whatever* probabilities might be associated with the constituents of the partition of Λ, the expected value of the quantity s defined in (1) would yield

$$E[A(\lambda,\mathbf{a})\, B(\lambda,\mathbf{b})] - E[A(\lambda,\mathbf{a})\, B(\lambda,\mathbf{b}')] + E[A(\lambda,\mathbf{a}')\, B(\lambda,\mathbf{b})] + E[A(\lambda,\mathbf{a}')\, B(\lambda,\mathbf{b}')] \quad (5)$$

on account of the linearity of an expectation operator. Expectation is understood to be evaluated with respect to some distribution over $\lambda \in \Lambda$, which admits a density $\rho(\lambda)$. Evaluated over the constituents of the 16-component partition of Λ it would generate a probability mass function, i.e., a schedule of probabilities for the components of the partition of Λ that would derive from the density.

However, the upshot of the recognized functional relation is that we would have to write this expectation Equation (5) either as

$$E[s(\lambda)] \;=\; E[A(\mathbf{a})B(\mathbf{b})] \;-\; E[A(\mathbf{a})B(\mathbf{b}')] \;+\; E[A(\mathbf{a}')B(\mathbf{b})] \;+\; E\{G[A(\mathbf{a})B(\mathbf{b}), A(\mathbf{a})B(\mathbf{b}'), A(\mathbf{a}')B(\mathbf{b})],$$

or equivalently using a different one of the four functional relations among the products. On account of these functional relations among polarization products in a gedankenexperiment, quantum theory can only specify the expectations of any three products appearing in (5) and use optimization methods to determine the bounds these would imply on assessment of the fourth.

Equation (5) mimics that of Aspect's equation numbered (21), which he labels as his quantity S. This expectation must lie within $[-2, +2]$, the convex hull of the realm of possibility for the quantity he labels as s. This specifies Bell's inequality in this context: $-2 \le E(s) \le +2$.

Arthur Fine [6] had already recognized the reparameterization of the expectation $E(s)$ that could be afforded by an appropriate partition of the space Λ. It was only his understanding such a characterization of the problem *to require* the specification of a *complete* distribution over the results of the incompatible observations composing the linear combination defining s, that distracted quantum theoretical considerations from its usefulness. For quantum theory currently avoids such a complete specification. This does not negate the validity of supplementary variable modeling as a thought. We now can understand that quantum theoretic specifications merely reduce the space of cohering distributions to a convex polytopic subspace of the full space spanned by the possible outcomes of the gedankenexperiment. We shall observe the relevance of this remark and derive its detail in our concluding Section 4.

Let us now examine expectation (5) in more detail, studying specifically its first summand expectation, which will be paradigmatic of each. Evaluating expectation with respect to any quantum theoretic mass function appropriate to the partition $\{\Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+}, \Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}-}, \Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}+}, \Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}-}\}$ would yield

$$\begin{aligned} E[A(\lambda,\mathbf{a})B(\lambda,\mathbf{b})] &= (+1)P[\Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+}] \;+\; (-1)P[\Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}-}] \;+\; (-1)P[\Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}+}] \;+\; (+1)P[\Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}-}] \\ &= P[A(a)B(b) = +1] \;-\; P[A(a)B(b) = -1] \\ &= 2\, P[A(a)B(b) = +1] \;-\; 1 \\ &= 2\,\{\, P[\Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+}] \;+\; P[\Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}-}]\,\} \;-\; 1 \end{aligned} \quad (6)$$

That second equality holds because quantum probabilities respect the equalities $P_{++} = P_{--}$ and $P_{-+} = P_{+-}$. Now, the same form of the concluding representation would apply if either or both of $\mathbf{a}$ and $\mathbf{b}$ were replaced in these lines by $\mathbf{a}'$ or $\mathbf{b}'$, respectively. Recognizing that these developments rely on the supposition of local realism, we have completed our construction of the hidden variables setup. We are ready for an analysis of $E[s(\lambda)]$ with respect to any distribution in the cohering QM polytope over the sixteen constituents that partition Λ.

We conclude this introduction to the hidden variables interpretation of spin expectations by noting that the representations of Equation (6) do not preclude *any* coherent expectations of the products of spins at A and B whatsoever, just so long as they respect the symmetry conditions involving $P_{-+} = P_{+-}$ and $P_{++} = P_{--}$. It would be useful for representing both the standard QM-motivated probabilities as well as the expectations Aspect refers to as the "naive supplementary model", for examples. Both of these structures of spin expectations can be represented by hidden variables parameterizations. Because this understanding conflicts with that of Aspect, who thought that QM probabilities cannot be parameterized by hidden variables while his proposed "naive model" can, we should defer our analysis for a while to dwell briefly on this thought.

2.2. On the Substantive Content of an HV Parameterization

"Hidden variables" motivations for an assessment of uncertainty about any quantity are merely considerations that identify a specific reparameterization of the quantity in question. They might be helpful in someone's assessment of the expectation, or they might not. They are surely not *required* for the assertion of relevant probabilities. Identical probability assertions regarding the observable quantities might be promoted both by someone who thinks about an experimental situation in terms of hidden variables and by someone who does not. This is the content of Equation (6). The two viewpoints are equivalent observationally, for as of now the hidden variables are, of course, hidden.

Mathematically, supplementary variables theory amounts merely to a 1-1 transformation of the partition of the space of numerical possibilities for the vector of quantities that specifies s, that is, $[\, A(\lambda, \mathbf{a})\, B(\lambda, \mathbf{b}),\ A(\lambda, \mathbf{a})\, B(\lambda, \mathbf{b}'),\ A(\lambda, \mathbf{a}')\, B(\lambda, \mathbf{b}),\ A(\lambda, \mathbf{a}')\, B(\lambda, \mathbf{b}')\,]$, to a different partition, the 16-constituent partition of Λ which was defined in Equation (4). The former partition of the possible observation values contains the sixteen (4×1) vectors whose components each equal either -1 or $+1$; and each of these vectors contains either none, one, two, three or four -1's, in any order. The latter partition is a partition of the hidden variables space, Λ. The probabilities one might assess for the observation vector partition must be the same as those one would assess for the hidden variables. Of course, one cannot directly assess the probabilities for the hidden variables, because they are hidden. Einstein's proposition merely imagined them as explanations of his viewpoint that the theory of quantum mechanics must be incomplete. The probabilities it specifies derive merely from the characterization of symmetries it identifies in studied conditions. Nonetheless, the probabilities he would assert for the observable spin values while in ignorance of such hidden variables are identical to the probabilities of those who imagine that the QM probabilities are actually inherent in the photons.

The bottom line is that a hidden variables explanation of polarization product expectations can apply to any probability distribution for the possible observation values whatsoever, including those motivated by the theory of quantum mechanics. Aspect's understanding that quantum probabilities are incompatible with supplementary variables is simply mistaken. Let's get down to the business of assessing the general form of an expectation for the Aspect/CHSH/Bell quantity, s, and particularly the expectation motivated by quantum theory in keeping with an imagination of supplementary variables.

2.3. Assessing $E(s)$ in the Situation of Entangled Distributions for $(\mathbf{A}, \mathbf{B})$

We shall now address the assessment of $E(s)$, and in particular a surprising identification of $E_{QM}(s)$ which challenges once again the Aspect/Bell assertion that $E_{QM}(s) = 2$. Firsly, recall that

$$E[s(\lambda)] \ = \ E[A(\lambda, \mathbf{a})\, B(\lambda, \mathbf{b})] - E[A(\lambda, \mathbf{a})\, B(\lambda, \mathbf{b}')] + E[A(\lambda, \mathbf{a}')\, B(\lambda, \mathbf{b})] + E[A(\lambda, \mathbf{a}')\, B(\lambda, \mathbf{b}')]\,.$$

Then using the form of Equation (6) for each of these four product summands, viz.,

$$E[A(\lambda,\mathbf{a})B(\lambda,\mathbf{b})] \;=\; 2\,\{P[\Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+}] \;+\; P[\Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}-}]\} \;-\; 1$$

and performing the summations (with the one difference understood) appropriate to the definition of s, we can write

$$\begin{aligned} E[s(\lambda)] = \;\; & 2\,\{\, P[\Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+}] \;+\; P[\Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}-}] \\ & +\; P[\Lambda_{\mathbf{a}'+}\Lambda_{\mathbf{b}+}] \;+\; P[\Lambda_{\mathbf{a}'-}\Lambda_{\mathbf{b}-}] \\ & +\; P[\Lambda_{\mathbf{a}'+}\Lambda_{\mathbf{b}'+}] \;+\; P[\Lambda_{\mathbf{a}'-}\Lambda_{\mathbf{b}'-}] \\ & -\; P[\Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}'+}] \;-\; P[\Lambda_{\mathbf{a}-}\Lambda_{\mathbf{b}'-}]\,\} \;-\; 2 \end{aligned} \tag{7}$$

Now, our next insight will look quite messy, algebraically, but what we need to do is to register the fact that probabilities for these various partition constituents are necessarily related to one another. Each of them arises from assessing uncertainty regarding the sum of four constituents of the 16-constituent partition of Λ we detailed in Equation (4). For each of the probability summands appearing in Equation (7), we need to identify which components of the 16-component partition of Λ require a probability assessment. For example, the first partition constituent $\Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+}$, which is assessed with the first probability appearing in (7), is composed of the union of four constituents of the finer partition of Λ, viz.,

$$\Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+} \;=\; \Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+}\Lambda_{\mathbf{a}'+}\Lambda_{\mathbf{b}'+} \,\cup\, \Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+}\Lambda_{\mathbf{a}'-}\Lambda_{\mathbf{b}'+} \,\cup\, \Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+}\Lambda_{\mathbf{a}'+}\Lambda_{\mathbf{b}'-} \,\cup\, \Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+}\Lambda_{\mathbf{a}'-}\Lambda_{\mathbf{b}'-}$$

These are the constituents we agreed to number $1,2,3$ and 4.

Thus, $P[\Lambda_{\mathbf{a}+}\Lambda_{\mathbf{b}+}]$ must be accorded with the sum of their probabilities. In order to clinch the messy algebraic implications of this recognition as it pertains to the other partition components of Λ, I present the complete composition of $E[s(\lambda)]$ below, using a schematic format. The three blocks of summations headed with a "+" in their top left-hand corners correspond to the pairs of summands in the first three rows of the display of (6), and the final block headed with a $-$ corresponds to the final row which involves a subtraction. The four-plex of partitions in columns that follow each partition component in the top left of a half-block designate the conjoined rarifications of that component, which together complete its identification via the terms displayed in Equation (4). At the left or right of each such refined term appears the component number of Equation (4) that it represents.

$+$							
	*1	$\Lambda_{\mathbf{a}+}\,\Lambda_{\mathbf{b}+}$	$\Lambda_{\mathbf{a}'+}\,\Lambda_{\mathbf{b}'+}$	$+$	$\Lambda_{\mathbf{a}-}\,\Lambda_{\mathbf{b}-}$	$\Lambda_{\mathbf{a}'+}\,\Lambda_{\mathbf{b}'+}$	*13
	*2		$\Lambda_{\mathbf{a}'+}\,\Lambda_{\mathbf{b}'-}$	$+$		$\Lambda_{\mathbf{a}'+}\,\Lambda_{\mathbf{b}'-}$	*14
	*3		$\Lambda_{\mathbf{a}'-}\,\Lambda_{\mathbf{b}'+}$	$+$		$\Lambda_{\mathbf{a}'-}\,\Lambda_{\mathbf{b}'+}$	*15
	*4		$\Lambda_{\mathbf{a}'-}\,\Lambda_{\mathbf{b}'-}$	$+$		$\Lambda_{\mathbf{a}'-}\,\Lambda_{\mathbf{b}'-}$	*16
$+$							
	*1	$\Lambda_{\mathbf{a}'+}\,\Lambda_{\mathbf{b}+}$	$\Lambda_{\mathbf{a}+}\,\Lambda_{\mathbf{b}'+}$	$+$	$\Lambda_{\mathbf{a}'-}\,\Lambda_{\mathbf{b}-}$	$\Lambda_{\mathbf{a}+}\,\Lambda_{\mathbf{b}'+}$	*11
	*2		$\Lambda_{\mathbf{a}+}\,\Lambda_{\mathbf{b}'-}$	$+$		$\Lambda_{\mathbf{a}+}\,\Lambda_{\mathbf{b}'-}$	*12
	*5		$\Lambda_{\mathbf{a}-}\,\Lambda_{\mathbf{b}'+}$	$+$		$\Lambda_{\mathbf{a}-}\,\Lambda_{\mathbf{b}'+}$	*15
	*6		$\Lambda_{\mathbf{a}-}\,\Lambda_{\mathbf{b}'-}$	$+$		$\Lambda_{\mathbf{a}-}\,\Lambda_{\mathbf{b}'-}$	*16
$+$							
	*1	$\Lambda_{\mathbf{a}'+}\,\Lambda_{\mathbf{b}'+}$	$\Lambda_{\mathbf{a}+}\,\Lambda_{\mathbf{b}+}$	$+$	$\Lambda_{\mathbf{a}'-}\,\Lambda_{\mathbf{b}'-}$	$\Lambda_{\mathbf{a}+}\,\Lambda_{\mathbf{b}+}$	*4
	*9		$\Lambda_{\mathbf{a}+}\,\Lambda_{\mathbf{b}-}$	$+$		$\Lambda_{\mathbf{a}+}\,\Lambda_{\mathbf{b}-}$	*12
	*5		$\Lambda_{\mathbf{a}-}\,\Lambda_{\mathbf{b}+}$	$+$		$\Lambda_{\mathbf{a}-}\,\Lambda_{\mathbf{b}+}$	*8
	*13		$\Lambda_{\mathbf{a}-}\,\Lambda_{\mathbf{b}-}$	$+$		$\Lambda_{\mathbf{a}-}\,\Lambda_{\mathbf{b}-}$	*16
$-$							
	*1	$\Lambda_{\mathbf{a}+}\,\Lambda_{\mathbf{b}'+}$	$\Lambda_{\mathbf{a}'+}\,\Lambda_{\mathbf{b}+}$	$+$	$\Lambda_{\mathbf{a}-}\,\Lambda_{\mathbf{b}'-}$	$\Lambda_{\mathbf{a}'+}\,\Lambda_{\mathbf{b}+}$	*6
	*9		$\Lambda_{\mathbf{a}'+}\,\Lambda_{\mathbf{b}-}$	$+$		$\Lambda_{\mathbf{a}'+}\,\Lambda_{\mathbf{b}-}$	*14
	*3		$\Lambda_{\mathbf{a}'-}\,\Lambda_{\mathbf{b}+}$	$+$		$\Lambda_{\mathbf{a}'-}\,\Lambda_{\mathbf{b}+}$	*8
	*11		$\Lambda_{\mathbf{a}'-}\,\Lambda_{\mathbf{b}-}$	$+$		$\Lambda_{\mathbf{a}'-}\,\Lambda_{\mathbf{b}-}$	*16

The expectation Equation (7) for $E(s)$ says firstly to sum the probabilities for the first three pairs of partition constituents headed by a plus sign, and then to subtract the probability for the fourth constituent pair headed by a minus. Then, double this result and subtract 2. On the right-hand and left-hand sides of each of the four banks displayed above appears an exhaustive list of constituent numbers from the 16-partition of Λ whose probabilities are to be summed (the first three banks) or subtracted (the last bank). Each constituent in the summable list is numbered. As can be seen there, each of the probabilities for constituents listed in the bottom bank (to be subtracted) will cancel the probability for a constituent matching it in one of the first three banks. As a result, the constituents remaining whose probabilities are to be summed are those numbered $1, 2, 4, 5, 12, 13, 15$ and 16; and each of these appears twice. Equation (7) says that to compute $E[s(\lambda)]$, this doubled sum should then itself be doubled, and finally have the number 2 subtracted. Because the sum of the probabilities (whatever values they might have) for the numbered constituents remaining is surely within $[0, 1]$, its double is surely within $[0, 2]$. Doubling that number will yield a number within $[0, 4]$, and subtracting 2 according to the directions of Equation (7), *will surely yield a number within the interval* $[-2, +2]$, just as required by coherency and just as required by Bell's inequality.

This numerical analysis makes more convincing than ever the conclusion that the Aspect/Bell derivation of $E_{QM}(s) = 2\sqrt{2}$ is incorrect, and that Bell's inequality is not defied at all. It matters not what might be the probabilities tendered regarding the components of s, whether based on quantum theory, on Aspect's caricature of naive realism, or whatever. Moreover, this result derives from the supposition of supplementary variables! Rather than being incongruous with the quantum theoretic assessment of $E(s)$, the proposition of hidden variables underlies the correct numerical assessment of this expectation as an interval lying within the bounds determined by Bell's inequality. A demonstration that the assessment motivated by quantum theory resolves only to an interval is the burden of our ultimate analysis, to which we now turn.

3. Computational Support for the Bounded Assessment of $E_{QM}(s)$

The reparameterization of the possible results of an imagined Aspect/CHSH/Bell experiment using the partition of Λ in our developments of Equations (1)–(3) was recognized long ago in the oft-mentioned article of Fine [6], which was contested and discussed in print [7]. The proposal of additional variables supplementary to the mere specification of the relative angles between the polarizers at stations A and B allows one to express quantum probabilities for the polarization observations in terms of probabilities for components of the partition of Λ. With this I surely agree. However, I believe Fine overstated the implication of his analysis when proposing (p 291) that "the existence of a deterministic hidden-variables model is strictly equivalent to the existence of a joint probability distribution function $P(AA'BB')$ for the four observables of the experiment, one that returns the probabilities of the experiment as marginals." Concluding my analysis of the situation in this Section, I shall now develop the bounding implications that the agreeable and powerful assertions of quantum mechanics actually do support regarding this matter. Identifying only a convex space of distributions that cohere with its propositions, these admittedly do not specify a complete joint distribution for the incompatible observations of the gedankenexperiment, nor for the widely touted incorrect expectation specification of $E_{QM}(s) = 2\sqrt{2}$. Nonetheless, they do provide for the formulation of a hidden variables model as we have provided herein.

To begin, the mere specification of the sixteen component partition of the hidden variables space Λ in Equation (4) does not presuppose the assertion of a probability distribution over these components. The analysis of Section 2.2 following this identification has determined that the quantum theoretic expectation $E(s)$ would require merely the specification of probabilities for a combination of components $1, 2, 4, 5, 12, 13, 15$, and 16 of this partition. Even at that, each of these probabilities would require an assertion regarding the joint outcome of four incommensurable experiments, and quantum theory explicitly avoids any such assertions. Where does this leave us?

Quantum theory is forthright, nonetheless, in specifying a precise expectation value for the polarization product of observations at *any one* of the paired angles at which an experiment might actually be conducted, $(\mathbf{a}, \mathbf{b})$, $(\mathbf{a}, \mathbf{b}')$, $(\mathbf{a}', \mathbf{b})$, and $(\mathbf{a}', \mathbf{b}')$. In the context of Aspect's experimental design, these relative angles between the polarizer directions at A and B are $-\pi/8, -3\pi/8, \pi/8$, and $-\pi/8$, respectively. In these settings, the well-known expectation values for polarization products in individual experiments are $E[A(\mathbf{a})B(\mathbf{b})] = E[A(\mathbf{a}')B(\mathbf{b})] = E[A(\mathbf{a}')B(\mathbf{b}')] = 1/\sqrt{2}$, and $E[A(\mathbf{a})B(\mathbf{b}')] = -1/\sqrt{2}$. Such details are well known from the exposition of Aspect (2002) and have been described by Lad [2], who provides further analysis to which we shall soon refer.

I now rely on your examination of the partition Equation (4) to recognize that the supplementary variables partition would correspond to a companion partition of the possible polarization product observation vectors in the gedankenexperiment, as shown in the columns of the following matrix:

$$
\begin{array}{c}
\Lambda_i \\
\,\,*\,*\,* \\
A(\mathbf{a}) \\
B(\mathbf{b}) \\
A(\mathbf{a}') \\
B(\mathbf{b}') \\
\,\,*\,*\,* \\
A(\mathbf{a})B(\mathbf{b}) \\
A(\mathbf{a})B(\mathbf{b}') \\
A(\mathbf{a}')B(\mathbf{b}) \\
A(\mathbf{a}')B(\mathbf{b}') \\
\,\,*\,*\,* \\
s(\lambda)
\end{array}
\quad
\left(\begin{array}{cccccccccccccccc}
\Lambda_1 & \Lambda_2 & \Lambda_3 & \Lambda_4 & \Lambda_5 & \Lambda_6 & \Lambda_7 & \Lambda_8 & \Lambda_9 & \Lambda_{10} & \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} & \Lambda_{15} & \Lambda_{16} \\
 & & & & & & & & & & & & & & & \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
 & & & & & & & & & & & & & & & \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
 & & & & & & & & & & & & & & & \\
2 & 2 & -2 & 2 & 2 & -2 & -2 & -2 & -2 & -2 & -2 & 2 & 2 & -2 & 2 & 2
\end{array}\right).
$$

Columns of the first four numerical rows of this matrix identify the possible polarization observations of the four components of the gedankenexperiment, and those of the second four rows represent the products of one of the possible A's with one of the B's. The final row of the matrix displays that the value of $s(\lambda)$ associated with any of its possible columns can equal only $+2$ or -2. It should be noticed that while the first four rows of this matrix exhibit sixteen distinct columns, the second four rows contain only eight distinct columns. The last eight columns of possible polarization products in these rows repeat the first eight in reversed order. This alerts us to the fact that the value of any one of the those rows is related functionally to the other three via a mapping $\{-1, +1\}^3 \to \{-1, +1\}$. The first eight columns of the first three rows in this group exhaust the possibilities of the cartesian product $\{-1, +1\}^3$. This would constitute the domain of the function we designated as $G(\cdot, \cdot, \cdot)$ earlier in Section 2.1. In fact, this is true of *any three rows* of product possibilities chosen from the four. Any one of these four rows is restricted to equal a value restricted by the same function of the other three. *This set of symmetric functional relations among the possible polarization products composing the value of* $s(\lambda)$ *in a gedankenexperiment on a single pair of photons is what gives rise to the error in the widely presumed expectation value for* $E(s)$ *as* $2\sqrt{(2)}$. See [2] for details. For use in what follows, let's designate the four row vectors of polarization product values appearing in the second bank of this matrix by the names $\mathbf{r_{ab}}, \mathbf{r_{ab'}}, \mathbf{r_{a'b}}$, and $\mathbf{r_{a'b'}}$.

Of course, quantum theory does not specify a vector $\mathbf{q}_{16}$ of probabilities for the 16 column vectors of possible gedanken polarization outcomes. To do so would require a probability specification for the outcomes of unobservable joint experiments. While admitting this, nonetheless quantum theory does explicitly say something about some *specific convex combinations* of these unspecified probabilities. For each quantum expectation of a polarization product at a single angle setting specifies the value of a linear combination of them, viz., $E[A(\mathbf{a})B(\mathbf{b})] = \mathbf{r_{ab}}\mathbf{q}_{16}$; $E[A(\mathbf{a})B(\mathbf{b}')] = \mathbf{r_{ab'}}\mathbf{q}_{16}$; $E[A(\mathbf{a}')B(\mathbf{b})] = \mathbf{r_{a'b}}\mathbf{q}_{16}$; and

$E[A(\mathbf{a}')B(\mathbf{b}')] = \mathbf{r}_{\mathbf{a}'\mathbf{b}'}\mathbf{q}_{16}$. However, the functional relations among the four components of the CHSH/Bell quantity s imply that only three of these expectations can be recognized at a time when the gedankenexperiment is meant to be applied to the same two photons at all four of the angled polarization settings.

The upshot of these considerations is that the implications of quantum theory can be identified by the specification of four paired linear programming problems. I shall first specify one of the LP problems algebraically, and then describe it. Here is one of the LP pairs:

Find the minimum and maximum values of $2\,\{[\,2\,(1\,1\,0\,1\,1\,0\,0\,0\,0\,0\,0\,1\,1\,0\,1\,1)\,\mathbf{q}_{16}]\} - 2,$

$$\text{subject to the linear restrictions} \quad \begin{pmatrix} \mathbf{r}_{\mathbf{ab}} \\ \mathbf{r}_{\mathbf{ab}'} \\ \mathbf{r}_{\mathbf{a}'\mathbf{b}} \end{pmatrix} \mathbf{q}_{16} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix},$$

as required of the expectations that we have presumed to be specified,

and where the components of $\mathbf{q}_{16}$ must be non-negative and must sum to 1.

The other three pairs of min/max LP routines would derive from changing the choice of the three $\mathbf{r}_{\mathbf{a}^*\mathbf{b}^*}$ vectors specifying the linear constraints for the other three possibilities.

The objective function vector $(1\,1\,0\,1\,1\,0\,0\,0\,0\,0\,0\,1\,1\,0\,1\,1)$ identifies the components of $\mathbf{q}_{16}$ involved in the representation of $E(s)$ as the sum of component numbers 1, 2, 4, 5, 12, 13, 15, and 16, as we recognized in the discussion following the schematic formulation above. Appropriately, the objective function says to double this sum, then double it again and subtract 2. The three linear constraining equations represent the restrictions required by the quantum theoretic specification of expectations for the polarization products in three of the angle settings. I shall not be more expansive in the description of this formulation here, because I have provided complete ornate details of this type of computational argument in my exposition [2] of the error involved in the Aspect/Bell problem. This can be read for further clarity if the explanation here is problematic.

While the detailed algebraic format of the computations described here are different from those described in the Aspect/Bell analysis, the problem the two formats resolve is precisely the same problem. It is not surprising that the numerical solutions to the eight sets of linear programming problems yield identical results for the extreme solution values of the vector $\mathbf{q}_8$. The LP computations find that quantum theoretic probabilities require only an expectation assessment of $E(s)$ within a (rounded) interval $(1.1213, 2]$, rather than that it equal the fabled $2\sqrt{2}$, which is mistaken. This is despite the fact that the formulation produced here relies explicitly on a characterization of hidden variables, while the other does not. Details of the four-dimensional polytopic resolution of the problem can be viewed in [2].

4. Concluding Remarks

Hardly impossible at all, we have formalized a mathematical structure for the supplementary variable explanation of the results of theoretical quantum physics pertinent to the Aspect/Bell problem. This construction defies the extended claims of quantum theorists who have generated a literature of results proclaiming its impossibility, categorized as a class of "no go theorems" [8,9]. To be sure, our construction pertains to a specific case of a two-dimensional problem. However, published protestations that this cannot be done in a three or four dimensional problem drag in a red herring. The higher dimensions involved in expanding the construction to the more complicated design considered by Greenberger et al. [10] or even the 3×3 spin problem discussed by Mermin [11] prove no hindrance to formalizing more complex partitions of the ensemble of possibilities for the quantity observations they allow. Mathematical detail for this claim can be found in [12,13].

In any such problem, quantum probabilities are derived for quantity observation possibilities that appear as eigenvalues of specific Hermitian operators acting on some Hilbert space of states. Formally, the implication of the hidden variables explanation of a quantum probability problem is that there can be further as-yet-unstipulated operators acting on the Hilbert space of an embellished state variable which might or might not commute with the operators whose stochastics are already prescribed by the theory. If they ever were identifiable operationally, their observation values would provide a functional basis for forecasting quantum phenomena with greater precision. The specification of such operators could be posed as an abstract mathematical problem. It would be a problem of scientific physics to imagine and design an observable physical process whose engagement could be codified by such operators. Until such progress might be made in experimental research physics, the current theory can well be considered to be incomplete.

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