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Accumulators and Bookmaker's Capital with Perturbed Stochastic Processes

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Abstract: The sports betting industry has been growing at a phenomenal rate and has many similarities to the financial market in that a payout is made contingent on an outcome of an event. Despite this, there has been little to no mathematical focus on the potential ruin of bookmakers. In this paper, the expected profit of a bookmaker and probability of multiple soccer matches are observed via Dirac notations and Feynman's path calculations. Furthermore, we take the unforeseen circumstances into account by subjecting the betting process to more uncertainty. A perturbed betting process, set by modifying the conventional stochastic process, is handled to scale and manage this uncertainty.

Keywords: betting; path calculation; capital modelling; bookmakers; accumulators; multiples; ACCAs

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1. Introduction

The gambling industry is a significantly large industry and has been growing its online presence consistently over the past years. The estimated amount of gross gambling revenue, being amount wagered less payouts, for online gambling businesses in UK and EU was around twenty-six billion euro during 2020 [1]. Around 40% of this is is generated from sports betting. The growth in online gambling is bound to continue as more markets are opening up since gambling licences are being generally viewed as a means to increase revenue for states [2]. It is therefore fairly straightforward to note that online sports betting is a multi-billion dollar industry.

Sports betting has therefore been a focus of interest in different academic circles with significant emphasis on problem gambling (for example [3,4]); profiling gamblers (for example [5,6]); the inefficient functioning of betting markets (for example [7–9]); or strategies to beat markets (for example [10,11]). In that respect, the focus has been on the clients of these services or the actual numbers/predictions generated by the services.

It would be expected that the solvency of a multi-billion dollar industry would also be of great interest to academics and regulators. Yet there has been scarce research on the solvency of the providers of this service, the sports betting companies themselves, with notable exceptions being [12,13]. Ref. [12] introduces a theoretical quantitative framework that can be used to measure the potential risk of a sports betting company from its trading operations by subdividing its portfolio of wagers into bundles according to their likelihood size. Ref. [13] is an in-depth overview and comparison of the perceived risks faced by banks and gambling companies ranging from compliance to operational risks.

The rationale for this vacuum in bookmaker solvency could be due to bookmakers being considered as the 'bad' guys that do not contribute to the welfare of society and the lack of systemic effects should a bookmaker fail. However, bookmakers are large corporations whose ruin, or inefficiencies, are bound to affect financial markets. The current state of affairs with respect to reserving is leading to an inefficient use of capital. Moreover, it would be ideal to have fair, stable, strong bookmakers that are able to enforce responsible gaming and limit access to money-laundering rather than bookmakers that are inefficient in their processes.

In this paper, we introduce the use of perturbed betting processes to determine bookmaker's capital more accurately and the likelihood of ruin by taking example of a series of accumulators (also known as multiples) on soccer matches. The results can be extended to other sports or events—we used soccer as it is the most popular sport and one of the few sports in which three outcomes are possible due to the prevalence of draws.

In addition to traditional representations and calculations, the paper focuses on Dirac notations and Feynman's path calculations in betting market scenarios. Although there is a vast literature on the use of these techniques applied in finance (see [14–17]), Dirac notations and Feyman's path calculations are not commonly applied in insurance and betting industries. Utev and Tamturk applied them into actuarial risk and capital modeling [18,19]. In follow-up joint papers, the approach was used for insurance claim analysis [20], catastrophic modelling by combining epidemic and actuarial models [21] and stock market data analysis [22].

2. Bookmaker's Expected Profit via Matrix Representation

A soccer match can end in one of three outcomes: home team win, draw and away team win. Let us assume that the probabilities for each one of these three outcomes are p_1, p_2 and p_3. However, bookmakers would use inflated probabilities when determining odds in order to guarantee a profit. We use standard notation as described in [11,23] to define implied probabilities a $\pi_i = p_i$ for $k > 0$ and $i = 1, 2, 3$.

European odds are defined by $e_i = 1/\pi_i$. Let w_i be the total value in units of wagers placed on outcome i. By all definitions, in terms of betting company, the revenue is $\sum_{i=1}^{3} w_i$ and expected payout is $\sum_{i=1}^{3} w_i p_i e_i$. Hence, the profit of the company can be defined simply for a single match by

$$E[C] = \sum_{i=1}^{3} w_i - \sum_{i=1}^{3} w_i p_i e_i.$$

When n soccer matches are considered, we have 3^n different scenarios (tickets) because there are 3 options in a match as home team win, draw and away team win as seen in Figure 1.

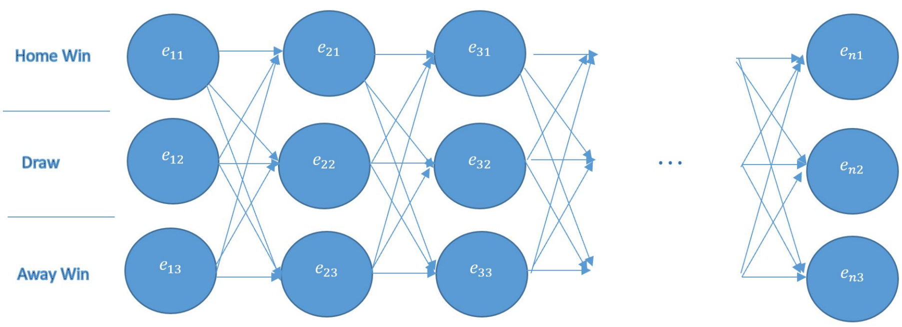

Figure 1. Possible paths for n matches.

The betting of one stake based on multiple outcomes is called a multiple or an accumulator. Accumulators allow bettors to place one small stake to potentially large returns but exponentially low likelihoods. Bettors can put money on 3^n different options for n matches under assumption that they bet with n matches. By default, the expected profit for n matches is defined by

$$E[C(n)] = \sum_{i=1}^{3^n} w_i - \sum_{i=1}^{3^n} w_i p_i e_i, \tag{1}$$

where p_i is probability of being n independent events, so $p_i \in \{p_{1j_1}p_{2j_2}...p_{nj_n}\}$ for $j_1, j_2, ...j_n \in \{1,2,3\}$. Notice that

$$\sum_{j_1=1}^{3}\sum_{j_2=1}^{3}...\sum_{j_n=1}^{3} p_{1j_1}p_{2j_2}...p_{nj_n} = 1.$$

Similarly, e_i representing multiply of n European odds on each match, and $e_i \in \{e_{1j_1}e_{2j_2}...e_{nj_n}\}$ for $j_1, j_2, ...j_n \in \{1,2,3\}$.

By default,

$$\sum_{j_1=1}^{3}\sum_{j_2=1}^{3}...\sum_{j_n=1}^{3} e_{1j_1}e_{2j_2}...e_{nj_n} = \sum_{j_1=1}^{3}\sum_{j_2=1}^{3}...\sum_{j_n=1}^{3} \frac{1}{p_{1j_1}p_{2j_2}...p_{nj_n}(1+k)^n}. \tag{2}$$

Bettors do not have to select to bet on every match. If the total number of matches is n, bettors can create tickets by selecting $n, n-1, n-2, ..., 3, 2$ or 1 matches as desired, so the total number of options is

$$\frac{n!}{n!0!}3^n + \frac{n!}{(n-1)!1!}3^{n-1} + \frac{n!}{(n-2)!2!}3^{n-2} + ... + \frac{n!}{1!(n-1)!}3^1.$$

In this circumstance, the formula of the expected capital of betting company can be modified by

$$E[C(n^*)] = \sum_{l=1}^{n}\sum_{j=1}^{\frac{n!}{l!(n-l)!}}\sum_{i=1}^{3^j} w_{i,l,j} - \sum_{l=1}^{n}\sum_{j=1}^{\frac{n!}{l!(n-l)!}}\sum_{i=1}^{3^j} w_{i,l,j}p_{i,l,j}e_{i,l,j}. \tag{3}$$

However, for simplicity, we will continue with the formula in (1) instead of the formula in (3).

From (1) and (2), the expected profit is

$$E[C(n)] = \frac{(1+k)^n - 1}{(1+k)^n}\sum_{i=1}^{3^n} w_i. \tag{4}$$

The company is expected to profit if $k > 0$ while negative values of k can cause ruin of the company [12]. However, the optimisation of k_i plays a vital role in terms of profitability of the company. In the case of a variable k_i, the formula in (4) can be modified as

$$E[C(n)] = \frac{((1+k_{1j_1})(1+k_{2j_2})...(1+k_{nj_n})) - 1}{(1+k_{1j_1})(1+k_{2j_2})...(1+k_{nj_n})}\sum_{i=1}^{3^n} w_i \quad \text{for} \quad j_1, j_2, ...j_n \in \{1,2,3\}. \tag{5}$$

We assume that the bookmaker has data to compute of $p_{i,j}$ and can control $e_{i,j}$ by k. Even though p_{i_j} is computable based on historical data, error term in the computation can produce surprising results.

As mentioned before, let w_i be the total value in units of wagers for outcome i. Then, $3^n \times 3^n$ matrix of wagers values

$$W = \begin{pmatrix} w_1 & 0 & 0 & \cdots & 0 \\ 0 & w_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & w_{3^n} \end{pmatrix}.$$

Let $P(i) = \begin{pmatrix} p_{i1} & 0 & 0 \\ 0 & p_{i2} & 0 \\ 0 & 0 & p_{i3} \end{pmatrix}$ and $E(i) = \begin{pmatrix} e_{i1} & 0 & 0 \\ 0 & e_{2i} & 0 \\ 0 & 0 & e_{3i} \end{pmatrix}$ be a probability matrix and European odds matrix respectively for ith match. Then, probability matrix P for n matches is defined via tensor product

$$P = P(1) \otimes P(2) \otimes \ldots \otimes P(n) = \begin{pmatrix} p_{11}p_{21}\ldots p_{n1} & 0 & 0 & \cdots & 0 \\ 0 & p_{11}p_{21}\ldots p_{n2} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & p_{13}p_{23}\ldots p_{n3} \end{pmatrix}.$$

Similarly,

$$E = E(1) \otimes E(2) \otimes \ldots \otimes E(n) = \begin{pmatrix} e_{11}e_{21}\ldots e_{n1} & 0 & 0 & \cdots & 0 \\ 0 & e_{11}e_{21}\ldots e_{n2} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & e_{13}e_{23}\ldots e_{n3} \end{pmatrix}.$$

One can note that P and E are $3^n \times 3^n$ dimensional matrices. Expected profit of the betting company is shown via a trace function representing sum of the diagonal elements of $W - WPE$.

$$E[C(n)] = tr(W - WPE).$$

3. Taking More Uncertainty into Account

In betting modeling, analysis of historical data does not reflect current situation perfectly. For example; referee mistakes [24], weather on the day of the match [25], distance travelled to match [26] and injuries of players can directly affect the outcome of the match. Therefore, even if a precise prediction is made, we need to modify the stochastic process with random uncertainty to make more realistic. This issue will be discussed more comprehensively in Section 5.

As mentioned before, for the ith match, probability of home win is $p_{i,1}$, probability of draw is $p_{i,2}$ and probability of away win is $p_{i,3}$. Let ϵ_i be probabilities of unpredictable things, and $\overline{p_{i,j}}$ are modified probabilities by ϵ_i as

$$\overline{p_{i,1}} = p_{i,1} + \epsilon_{i,1}$$
$$\overline{p_{i,2}} = p_{i,2} + \epsilon_{i,2}$$
$$\overline{p_{i,3}} = p_{i,3} + \epsilon_{i,3},$$

where we assume that $\epsilon_{i,1} + \epsilon_{i,2} + \epsilon_{i,3} = 0$ and $|\epsilon_{i,j}| < p_{i,j}$ for $j = 1, 2, 3$. Under these conditions, the corresponding matrix for the ith match can be written as

$$\begin{pmatrix} \overline{p_{i1}} & 0 & 0 \\ 0 & \overline{p_{i2}} & 0 \\ 0 & 0 & \overline{p_{i3}} \end{pmatrix} = \begin{pmatrix} p_{i1} & 0 & 0 \\ 0 & p_{i2} & 0 \\ 0 & 0 & p_{i3} \end{pmatrix} \begin{pmatrix} 1 + \frac{\epsilon_1}{p_{i1}} & 0 & 0 \\ 0 & 1 + \frac{\epsilon_2}{p_{i2}} & 0 \\ 0 & 0 & 1 + \frac{\epsilon_3}{p_{i3}} \end{pmatrix}.$$

Modified probabilities for n matches can be shown in the following matrix via tensor operator

$$\overline{P} = \overline{P(1)} \otimes \overline{P(2)} \otimes ... \otimes \overline{P(n)} = \begin{pmatrix} \overline{p_{11}}\overline{p_{21}}...\overline{p_{n1}} & 0 & 0 & \cdots & 0 \\ 0 & \overline{p_{11}}\overline{p_{21}}...\overline{p_{n2}} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \overline{p_{13}}\overline{p_{23}}...\overline{p_{n3}} \end{pmatrix}.$$

According to ϵ_1, ϵ_2 and ϵ_3 random variables, we can obtain different results, and get the average value by using simulation methods like Monte Carlo.

$$E[C] = \frac{1}{M} \sum_{z=1}^{M} tr(W - W\overline{P}E),$$

where M is a big enough number.

Notice that even if we modified P, E was not exposed to change because E is only computed by historical data while $\overline{P}$ based on historical data analysis and unpredictable things in recent time.

4. Bookmaker's Profit via Path Calculation

The path calculation approach has been applied to many different disciplines. To apply the path calculation approach in gaming, we have to think of the outcome of each match as a location or value at a different time. For example, let us consider five betting tickets for four matches as per the Table 1 below.

Table 1. Betting Tickets.

Tickets	Matches			
	1st Match	**2nd Match**	**3rd Match**	**4th Match**
1. Ticket	1	2	1	X
2. Ticket	X	1	1	2
3. Ticket	1	X	2	1
4. Ticket	2	2	X	X
5. Ticket	X	1	X	2

In this table, $1, X, 2$ represent home win, draw and away win, respectively. Now, we can consider the betting process via path calculation as per Figure 2. For example, the first ticket assumed the home teams will win the first and third matches, the away team will win the second match, and the last match will end in a draw. To make the our approach more clear, the betting process can be imagined as a time series by considering the matches as time, and results as locations or values.

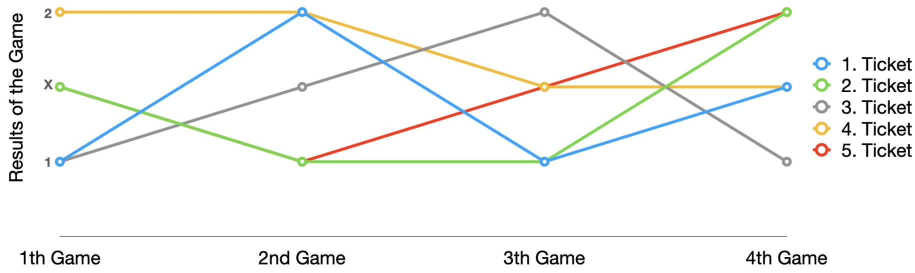

Figure 2. Path representation of Betting.

Between two spacetime points x_i at time t_i and x_{i+1} at time t_{i+1}, the propagator of the quantum system can be defined as the probability transition amplitude via the wavefunction [27]

$$K(x_i, t_i; x_{i+1}, t_{i+1}) = \langle \psi(x_i, t_i) | \psi(x_{i+1}, t_{i+1}) \rangle ,$$

where K is called as Feynman kernel. By using Hamiltonian operator with bra-ket notations, the propagator can be written as

$$\langle x_i | A(t_{i+1} - t_i) | x_{i+1} \rangle = \langle x_i | e^{(t_{i+1} - t_i)H} | x_{i+1} \rangle \tag{6}$$

In other words, a propagator makes the state function at spacetime point (x_i, t_i) move to another spacetime point (x_{i+1}, t_{i+1}).

Dirac defined notations about vectors in Hilbert space that corresponds to a physical system [28]. In Dirac formalism, $|x\rangle$ and $\langle x|$ represent column and row vectors, which are used to show quantum states. A bra is the Hermitian conjugate transpose of the corresponding ket. Elements of $\langle x_i|$ or $|x_i\rangle$ consist of 1 in position i and 0 elsewhere on computational basis. Surely, a ket can be defined in different ways (e.g., Hadamard and Circular basis [29]). A quantum operator like $A = \sum_{m,n} A_{m,n} |\psi_m\rangle \langle \psi_n|$ represents a matrix

$$\begin{pmatrix} A_{1,1} & A_{1,2} & \cdots \\ A_{2,1} & A_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

In the propagator in (6), $A = e^{-tH}$ is an operator in semigroup form with Hamiltonian operator H. In physics, the Hamiltonian of a system is defined as the sum of kinetics and potential energies [28],

$$H = K + V = \frac{-h}{2m} \bigtriangledown^2 + V,$$

where K and V are representing kinetic and potential energy of the system respectively.

Furthermore, Hamitonian operators provide a way to go to quantum formalism from a classical Markovian approach, which is characterized by the continuous semigroup $A(t) = e^{tQ}$ with generator Q. By taking minus generator operator equal to the corresponding Hamiltonian $(H = -Q)$, we have $A(t) = e^{-tH} = e^{tQ}$. In general, the Hamiltonian operator H does not need to be a generator, specifically for the cases of non-embeddable Markov chains [18].

Throughout the quantum parts of this paper, we work on a Hilbert Space that is a complete vector space. Regardless of finite or infinite dimensional, a Hilbert space is separable if its basis is countable [28].

The scalar product of the basis states $|x\rangle$ and $\langle x|$ is known as Dirac-delta function, and with the Fourier representation, it is written by [14]

$$\langle x|x'\rangle = \delta(x-x') = \int_{-\infty}^{+\infty} \frac{d\alpha}{2\pi} e^{i\alpha(x-x')} = \int_{-\infty}^{+\infty} \frac{d\alpha}{2\pi} \langle x|\alpha\rangle \langle\alpha|x'\rangle \,,$$

where $|\alpha\rangle$ is momentum space basis with the completeness equation $\int_{-\infty}^{+\infty} \frac{d\alpha}{2\pi} |\alpha\rangle \langle\alpha| = I$. Inner products of basis state and momentum basis state are

$$\langle x|\alpha\rangle = e^{i\alpha x}, \qquad \langle\alpha|x'\rangle = e^{-i\alpha x'},$$

where $|\alpha\rangle$ is the eigenstate of the momentum operator (e.g., let P be a momontum operator, so $P|\alpha\rangle = \alpha|\alpha\rangle$).

For actuarial random walks, the following propagator has been computed by Tamturk and Utev in the previous papers [18,19] and follow-up joint papers [20,21] with respect to several Hamiltonian operators with the help of the completeness equation and corresponding eigenvectors of different Hamiltonian operators,

$$\langle x| e^{-\Delta tH} |x'\rangle = \int_0^{2\pi} \frac{d\alpha}{2\pi} \langle x| e^{-\Delta tH} |\alpha\rangle \langle\alpha|x'\rangle = \frac{1}{2\pi}\int_0^{2\pi} (e^{ix\alpha}e^{-ix'\alpha})e^{-\Delta tK_\alpha} d\alpha, \tag{7}$$

where K_α and $|\alpha\rangle$ are eigenvalue and eigenvector of the Hamiltonian operator, i is complex unit.

4.1. Transition Probability via Path Calculation

Regarding the path calculations, the main difference between gambling and the other disciplines is that order of the games does not change the results. Let us consider the following propagator with a matrix operator by assuming that x_i is location (result) of i-th match as $x_i \in \{1,2,3\}$

$$\langle x_i| P_{i+1} |x_{i+1}\rangle = \langle x_i| \begin{pmatrix} p_{i+1,1} & p_{i+1,2} & p_{i+1,3} \\ p_{i+1,1} & p_{i+1,2} & p_{i+1,3} \\ p_{i+1,1} & p_{i+1,2} & p_{i+1,3} \end{pmatrix} |x_{i+1}\rangle \,.$$

Notice that all rows are taken as the same since we assume that the location of the $(i+1)$th match is independent from the ith match. Sum of probabilities of the all the possible paths can be showed by

$$\sum_{x_1=1}^{3}\sum_{x_2=1}^{3} \dots \sum_{x_n=1}^{3} \langle x_1| P_2 |x_2\rangle \langle x_2| P_3 |x_3\rangle \dots \langle x_{n-1}| P_n |x_n\rangle \langle x_n| P_1 |x_1\rangle = 1. \tag{8}$$

For the sake of simplicity, let us assume $\langle x_1| P_2 |x_2\rangle \langle x_2| P_3 |x_3\rangle \dots \langle x_{n-1}| P_n |x_n\rangle \langle x_n| P_1 |x_1\rangle$ denoted by $\langle \overline{x}| \overline{P} |\overline{x}\rangle$ because

$$\begin{aligned}
\langle \overline{x}| \overline{P} |\overline{x}\rangle &= \langle x_1| P_2 |x_2\rangle \langle x_2| P_3 |x_3\rangle \dots \langle x_{n-1}| P_n |x_n\rangle \langle x_n| P_1 |x_1\rangle \\
&= (\langle x_1| \otimes \langle x_2| \otimes \langle x_3| \otimes \dots \otimes \langle x_n|)(P_2 \otimes P_3 \otimes \dots \otimes P_n \otimes P_1)(|x_2\rangle \otimes |x_3\rangle \otimes \dots \otimes |x_n\rangle \otimes |x_1\rangle \\
&= (\langle x_1| \otimes \langle x_2| \otimes \langle x_3| \otimes \dots \otimes \langle x_n|)(P_1 \otimes P_2 \otimes P_3 \otimes \dots \otimes P_n)(|x_1\rangle \otimes |x_2\rangle \otimes |x_3\rangle \otimes \dots \otimes |x_n\rangle \\
&\quad \text{for the matrix operators with identical rows,} \\
&= \langle \overline{x}| \overline{P} |\overline{x}\rangle \,,
\end{aligned}$$

where $\overline{P} = P_1 \otimes P_2 \otimes P_3 \otimes \dots \otimes P_n$ and $|\overline{x}\rangle = |x_1\rangle \otimes |x_2\rangle \otimes |x_3\rangle \otimes \dots \otimes |x_n\rangle$. Note that this statement is not true for diagonal operators (P_i).

4.2. Hamiltonian Operator for Expected Profit

If an operator is self-adjoint (adjoint-operator is equal to itself), it can be called as Hermitian operator [28]. A Hermitian Hamiltonian matrix can be diagonalised with help of the following transformation

$$D = UHU^{-1},$$

where U is a unitary matrix. If U is a square matrix, $U^* = U^{-1}$ and $UU^* = U^*U = I$. Elements of D are eigenvalues representing energy levels.

$$P_i = U_1^* D_{1i} U_1 = U_1^* \begin{pmatrix} p_{i1} & 0 & 0 \\ 0 & p_{i2} & 0 \\ 0 & 0 & p_{i3} \end{pmatrix} U_1,$$

where U is a 3×3 unitary matrix, U^* is its adjoint. Similarly, W_i and E_i can be written in the following forms:

$$W_i = U_2^* D_{2i} U_2 = U_2^* \begin{pmatrix} e^{w_{i1}} & 0 & 0 \\ 0 & e^{w_{i2}} & 0 \\ 0 & 0 & e^{w_{i3}} \end{pmatrix} U_2$$

and

$$E_i = U_3^* D_{3i} U_3 = U_3^* \begin{pmatrix} e_{i1} & 0 & 0 \\ 0 & e_{i2} & 0 \\ 0 & 0 & e_{i3} \end{pmatrix} U_3.$$

Then,

$$\begin{aligned} P_i ln(W_i) E_i &= U_1^* D_{1i} U_1 ln(U_2^* D_{2i} U_2) U_3^* D_{3i} U_3 \\ &= U^* D U \\ &= U^* \begin{pmatrix} p_{i1} w_{i1} e_{i1} & 0 & 0 \\ 0 & p_{i2} w_{i2} e_{i2} & 0 \\ 0 & 0 & p_{i3} w_{i3} e_{i3} \end{pmatrix} U, \end{aligned}$$

where $U = U_1 U_2 U_3$ is 3×3 unitary matrix.

Hamiltonian operator can be defined for the ith match by

$$H_i = ln(W_i) - P_i ln(W_i) E_i.$$

However, H^n cannot be generalized for multimatches case by

$$H^n \neq ln(W_1 \otimes W_2 \otimes ... \otimes W_n) - (P_1 ln(W_1) E_1 \otimes P_2 ln(W_2) E_2 \otimes ... \otimes P_n ln(W_n) E_n)$$

because $ln(W_1 \otimes W_2 \otimes ... \otimes W_n)$ does not give the total wage amount for each possible result; so instead of that, the $3^n \times 3^n$ dimensional W matrix mentioned in Section 2 is taken into account.

$$\begin{aligned} H^n &= W - W(P_1 E_1 \otimes P_2 E_2 \otimes ... \otimes P_n E_n) \\ &= V^* \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{3^n} \end{pmatrix} V, \end{aligned}$$

where V is $3^n \times 3^n$ unitary matrix, and the diagonal elements can be computed as

$$
\begin{aligned}
d_1 &= (w_1) - w_1\Big(p_{11}e_{11}p_{21}e_{21}...p_{(n-1)1}e_{(n-1)1}p_{n1}e_{n1}\Big) \\
d_2 &= (w_2) - w_2\Big(p_{11}e_{11}p_{21}e_{21}...p_{(n-1)1}e_{(n-1)1}p_{n2}e_{n2}\Big) \\
d_3 &= (w_3) - w_3\Big(p_{11}e_{11}p_{21}e_{21}...p_{(n-1)1}e_{(n-1)1}p_{n3}e_{n3}\Big) \\
d_4 &= (w_4) - w_4\Big(p_{11}e_{11}p_{21}e_{21}...p_{(n-1)2}e_{(n-1)2}p_{n1}e_{n1}\Big) \\
d_5 &= (w_5) - w_5\Big(p_{11}e_{11}p_{21}e_{21}...p_{(n-1)2}e_{(n-1)2}p_{n2}e_{n2}\Big) \\
d_6 &= (w_6) - w_6\Big(p_{11}e_{11}p_{21}e_{21}...p_{(n-1)2}e_{(n-1)2}p_{n3}e_{n3}\Big) \\
d_7 &= (w_7) - w_7\Big(p_{11}e_{11}p_{21}e_{21}...p_{(n-1)3}e_{(n-1)3}p_{n1}e_{n1}\Big) \\
&\vdots \\
d_{3^n} &= (w_{3^n}) - w_{3^n}\Big(p_{13}e_{13}p_{23}e_{23}...p_{(n-1)3}e_{(n-1)3}p_{n3}e_{n3}\Big).
\end{aligned}
$$

Regarding finding the expected profit and capital of the betting company, we will use Hamiltonian operators H_p and H_e for transition probabilities and corresponding odds . Now let us find probability of a single match via the transformation in Equation (7).

$$
\begin{aligned}
\langle x_i| e^{-\Delta t H_p} |x_{i+1}\rangle &= \int_0^{2\pi} \frac{d\alpha}{2\pi} \langle x_i| e^{-\Delta t H_p} |\alpha\rangle \langle \alpha|x_{i+1}\rangle \\
&= \frac{1}{2\pi}\int_0^{2\pi} (e^{ix_i\alpha}e^{-ix_{i+1}\alpha})e^{-\Delta t K_\alpha} d\alpha,
\end{aligned} \tag{9}
$$

where the main complication is finding the eigenvalues of the Hamiltonian operators.

$$H_p |\alpha\rangle = K_\alpha |\alpha\rangle .$$

For a single match, we defined K_α as

$$K_\alpha = -ln(e^{i\alpha(1-x_1)}p_1 + e^{i\alpha(2-x_1)}p_2 + e^{i\alpha(3-x_1)}p_3),$$

where it can be noticed that we added x_1 inside of K_α to set up an independence between x_i and x_{i+1} .

$$e^{-\Delta t K_\alpha} = e^{i\alpha(1-x_1)}p_1 + e^{i\alpha(2-x_1)}p_2 + e^{i\alpha(3-x_1)}p_3.$$

When K_α is substituted into Equation (9), we will obtain

$$
\begin{aligned}
\langle x_i| e^{-\Delta t H_p} |x_{i+1}\rangle &= \frac{1}{2\pi}\int_0^{2\pi} (e^{ix_i\alpha}e^{-ix_{i+1}\alpha})e^{-\Delta t K_\alpha} d\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{i\alpha(x_i-x_{i+1})}(e^{i\alpha(1-x_i)}p_{i+1,1} + e^{i\alpha(2-x_i)}p_{i+1,2} + e^{i\alpha(3-x_i)}p_{i+1,3})d\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi} (e^{i\alpha(1-x_{i+1})}p_{i+1,1} + e^{i\alpha(2-x_{i+1})}p_{i+1,2} + e^{i\alpha(3-x_{i+1})}p_{i+1,3})d\alpha \\
&= \begin{cases} p_{i+1,1} & \text{if } x_{i+1} = 1 \\ p_{i+1,2} & \text{if } x_{i+1} = 2 \\ p_{i+1,3} & \text{if } x_{i+1} = 3 \\ 0 & \text{if } x_{i+1} \notin \{1,2,3\} \end{cases} .
\end{aligned} \tag{10}
$$

For n matches, probability of n events via path approach,

$$\langle x_1| e^{-\Delta t H_{p_2}} |x_2\rangle \langle x_2| e^{-\Delta t H_{p_3}} |x_3\rangle ... \langle x_{n-1}| e^{-\Delta t H_{p_n}} |x_n\rangle \langle x_n| e^{-\Delta t H_{p_1}} |x_1\rangle$$

can be computed by multiplying n integrals which will cause the long computation time, so for the sake of simplicity, we consider $\langle \overline{x}| e^{-tH_p^{\otimes n}} |\overline{x}\rangle$ for the n events, and define new K_α for n matches as

$$e^{-tK_\alpha} = e^{i\alpha(1-\overline{x})}P_{1,1} + e^{i\alpha(2-\overline{x})}P_{2,2} + e^{i\alpha(3-\overline{x})}P_{3,3} + ... + e^{i\alpha(3^n-\overline{x})}P_{3^n,3^n},$$

where $P = P_1 \otimes P_2 \otimes ... \otimes P_n$ and $|\overline{x}\rangle = |x_1\rangle \otimes |x_2\rangle \otimes |x_3\rangle \otimes ... \otimes |x_n\rangle$.

$$\begin{aligned} \langle x_1| e^{-\Delta t H_{p_2}} |x_2\rangle \langle x_2| e^{-\Delta t H_{p_3}} |x_3\rangle ... \langle x_{n-1}| e^{-\Delta t H_{p_n}} |x_n\rangle \langle x_n| e^{-\Delta t H_{p_1}} |x_1\rangle = \\ = \frac{1}{2\pi}\int_0^{2\pi} (e^{i\alpha(1-\overline{x})}P_{1,1} + e^{i\alpha(2-\overline{x})}P_{2,2} + e^{i\alpha(3-\overline{x})}P_{3,3} + \\ ... + e^{i\alpha(3^n-\overline{x})}P_{3^n,3^n})d\alpha \end{aligned} \tag{11}$$

$$= \begin{cases} p_{11}p_{21}...p_{n1} & \text{if } \overline{x} = 1 \\ p_{11}p_{21}...p_{n2} & \text{if } \overline{x} = 2 \\ p_{11}p_{21}...p_{n3} & \text{if } \overline{x} = 3 \\ \vdots & \\ p_{13}p_{23}...p_{n3} & \text{if } \overline{x} = 3^n \\ 0 & \text{if } \overline{x} \notin \{1,2,3,...,3^n\} \end{cases}. \tag{12}$$

Similarly, the odds for n matches can be computed by following path approach with Hamiltonian H_e^n operator

$$\begin{aligned} \langle \overline{x}| e^{-tH_e^n} |\overline{x}\rangle = \frac{1}{2\pi}\int_0^{2\pi} (e^{i\alpha(1-\overline{x})}E_{1,1} + e^{i\alpha(2-\overline{x})}E_{2,2} + e^{i\alpha(3-\overline{x})}E_{3,3} + \\ ... + e^{i\alpha(3^n-\overline{x})}E_{3^n,3^n})d\alpha. \end{aligned} \tag{13}$$

In this circumstance, the expected profit is

$$E[C(n)] = \sum_{\overline{x}=1}^{3^n}\Big(W_{\overline{x}} - W_{\overline{x}}\langle \overline{x}| e^{-H_p^n} |\overline{x}\rangle \langle \overline{x}| e^{-H_e^n} |\overline{x}\rangle\Big),$$

where $W_{\overline{x}}$ is $\overline{x}th$ diagonal element of W.

Example 1. *Let us assume the probabilities of three matches are as in Table 2, and let us calculate probability of being draw for the first match, away win for the second match and home win for the third match.*

Table 2. Probabilities of three matches in Premier League.

			Home Win	Draw	Away Win
Liverpool	vs.	Chelsea	0.4	0.3	0.3
Brighton	vs.	Arsenal	0.2	0.2	0.6
Leicester city	vs.	Fulham	0.7	0.2	0.1

In this circumstance, for draw, away win and home win, respectively,

$$|x_1\rangle = |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |x_2\rangle = |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |x_3\rangle = |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then,

$$
\begin{aligned}
|\overline{x}\rangle &= |x_1\rangle \otimes |x_2\rangle \otimes |x_3\rangle \\
&= |2\rangle \otimes |3\rangle \otimes |1\rangle \\
&= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= |16\rangle\,.
\end{aligned}
$$

$$
\begin{aligned}
P &= P_1 \otimes P_2 \otimes P_3 \\
&= \begin{pmatrix} 0.4 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{pmatrix} \otimes \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.6 \end{pmatrix} \otimes \begin{pmatrix} 0.7 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{pmatrix} \\
&= \begin{pmatrix}
0.056 \\
& 0.016 \\
&& 0.008 \\
&&& 0.056 \\
&&&& 0.016 \\
&&&&& 0.008 \\
&&&&&& 0.168 \\
&&&&&&& 0.048 \\
&&&&&&&& 0.024 \\
&&&&&&&&& 0.042 \\
&&&&&&&&&& 0.012 \\
&&&&&&&&&&& 0.006 \\
&&&&&&&&&&&& 0.042 \\
&&&&&&&&&&&&& 0.012 \\
&&&&&&&&&&&&&& 0.006 \\
&&&&&&&&&&&&&&& 0.126 \\
&&&&&&&&&&&&&&&& 0.036 \\
&&&&&&&&&&&&&&&&& 0.018 \\
&&&&&&&&&&&&&&&&&& 0.042 \\
&&&&&&&&&&&&&&&&&&& 0.012 \\
&&&&&&&&&&&&&&&&&&&& 0.006 \\
&&&&&&&&&&&&&&&&&&&&& 0.042 \\
&&&&&&&&&&&&&&&&&&&&&& 0.012 \\
&&&&&&&&&&&&&&&&&&&&&&& 0.006 \\
&&&&&&&&&&&&&&&&&&&&&&&& 0.126 \\
&&&&&&&&&&&&&&&&&&&&&&&&& 0.036 \\
&&&&&&&&&&&&&&&&&&&&&&&&&& 0.018
\end{pmatrix}.
\end{aligned}
$$

With P and $|x\rangle$, the probability of $\{draw, away\ win, home\ win\}$ respectively for the three matches is found by

$$
\begin{aligned}
\langle\overline{x}|\, e^{-\Delta t H_p}\, |\overline{x}\rangle &= \langle 16|\, e^{-H_p}\, |16\rangle \\
&= \frac{1}{2\pi}\int_0^{2\pi} (e^{i\alpha(1-\overline{x})}P_{1,1} + e^{i\alpha(2-\overline{x})}P_{2,2} + e^{i\alpha(3-\overline{x})}P_{3,3} + \\
&\quad ... + e^{i\alpha(3^n-\overline{x})}P_{3^n,3^n})d\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi} (e^{i\alpha(1-16)}0.056 + e^{i\alpha(2-16)}0.016 + e^{i\alpha(3-16)}0.008 + \\
&\quad ... + e^{i\alpha(27-16)}0.018)d\alpha \\
&= 0.126.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\langle 16|\, e^{-H_e}\, |16\rangle = \frac{1}{2\pi}\int_0^{2\pi} &(e^{i\alpha(1-16)}E_{1,1} + e^{i\alpha(2-16)}E_{2,2} + e^{i\alpha(3-16)}E_{3,3} + \\
&... + e^{i\alpha(27-16)}E_{27,27})d\alpha.
\end{aligned}
$$

The profit of the betting company can be computed as

$$
E[C(n)] = \sum_{\overline{x}=1}^{3^n} \Big(W_{\overline{x}} - W_{\overline{x}}\, \langle\overline{x}|\, e^{-H_p^n}\, |\overline{x}\rangle\, \langle\overline{x}|\, e^{-H_e^n}\, |\overline{x}\rangle \Big).
$$

4.3. Ruin Probability of the Betting Company

Let n_i, $i = 1, 2, ...m$ be numbers of the matches in the i-th week. Then, ruin probability of the company at the end of the m weeks with assumption of 0 initial reserve can be computed by

$$\begin{aligned} P(m \geqslant T) &= 1 - P(T > m) \\ &\simeq 1 - \Big(\sum_{i_1=1}^{\infty} P(E[C_1(n_1)] = i_1) \sum_{i_2=1-i_1}^{\infty} P(E[C_2(n_2)] = i_2) \cdots \\ &\sum_{i_m=1-(i_1+i_2+...+i_{m-1})}^{\infty} P(E[C_m(n_m)] = i_m) \Big), \end{aligned} \tag{14}$$

where T is the ruin time defined by

$$T = min\{t | C_1 + C_2 ... + C_t \leqslant 0 \quad \text{and} \quad C_1 + C_2 + ... + C_{t-1} > 0\}.$$

As seen from the ruin formula in (14), we consider weekly capital of the company as integer values—this can be done by truncated numerical approach. However, finding the probability of the capital is based on complex computations for large time scale and high number of matches. Another challenge in terms of the computation is dynamic odds. We refer this to future researches.

5. Perturbation of Betting Process

Classical Markov processes may be perturbed to produce new Markov chain with similar statistical properties [30,31]. Perturbation of Markov process is quite an important approach in order to take unpredictable things into account. Therefore, the logic is simply based on modification of the random process without changing the general properties.

In continuous time, the transition matrix can be found via the generator matrix.

$$A(0) = \lim_{t \to 0} A(t) = I,$$

$$A'(0) = \lim_{\varepsilon \to 0} \frac{A(\varepsilon) - I}{\varepsilon} = Q,$$

where Q is called the generator of the continuous time Markov process

$$Q = \begin{pmatrix} q_{0,0} & q_{0,1} & q_{0,2} & q_{0,3} & q_{0,4} & q_{0,5} & \cdots \\ q_{1,0} & q_{1,1} & q_{1,2} & q_{1,3} & q_{1,4} & q_{1,5} & \cdots \\ q_{2,0} & q_{2,1} & q_{2,2} & q_{2,3} & q_{2,4} & q_{2,5} & \cdots \\ q_{3,0} & q_{3,1} & q_{3,2} & q_{3,3} & q_{3,4} & q_{3,5} & \cdots \\ q_{4,0} & q_{4,1} & q_{4,2} & q_{4,3} & q_{4,4} & q_{4,5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{15}$$

The sum of the elements in each row of Q is zero with

$$\sum_{j=1, j \neq i} q_{i,j} = -q_{i,i},$$

where

$$q_{i,j} = \lim_{\Delta t \to 0} \frac{A_{i,j}(\Delta t)}{\Delta t} \geq 0 \quad \text{and} \quad q_{i,i} \leq 0.$$

For the small Δt,

$$\begin{aligned} A_{i,j} &= q_{i,j} \Delta t + O(\Delta t) \quad \text{for} \quad i \neq j \\ A_{i,i} &= 1 + q_{i,i} \Delta t + O(\Delta t). \end{aligned}$$

As mentioned in Section 4, $A(\Delta t) = e^{-\Delta t H} = e^{\Delta t Q}$, so the stochastic process can be perturbed by modifying the generator matrix. For example, let us consider 2×2 generator matrix on $\mathbb{C}^2$

$$Q = \begin{pmatrix} q_{0,0} & q_{0,1} \\ q_{1,0} & q_{1,1} \end{pmatrix}. \tag{16}$$

Then, as mentioned in Feigin and Rubinstein's article [30], perturbed generator Q^* can be found by

$$\begin{aligned} Q^* &= X_I Q + (I - X_I) Q R \\ &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q_{0,0} & q_{0,1} \\ q_{1,0} & q_{1,1} \end{pmatrix} + (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}) \begin{pmatrix} q_{0,0} & q_{0,1} \\ q_{1,0} & q_{1,1} \end{pmatrix} R \\ &= \begin{pmatrix} q_{0,0} & q_{0,1} \\ 0 & q_{1,1} + q_{1,0} R_{0,1} \end{pmatrix} \quad \text{for} \quad R = \begin{pmatrix} 0 & R_{0,1} \\ 0 & I \end{pmatrix}, \end{aligned}$$

where R is a replacement matrix. With different versions of the replacement function, we can modify the stochastic process in various ways. However, we follow a different way. As mentioned in Section 4 , Hamiltonian operator H does not need to be $-Q$, which us gives more flexibility. The perturbed Hamiltonian operator can be defined by

$$H_\epsilon = H_p + \mathcal{E} V,$$

where V is a Hermitian operator. For the ith match, the probabilities of the changes caused unpredictable situations is displayed by

$$\mathcal{E}(i) = \begin{pmatrix} 1 + \frac{\epsilon_{i1}}{p_{i1}} & 0 & 0 \\ 0 & 1 + \frac{\epsilon_{i2}}{p_{i2}} & 0 \\ 0 & 0 & 1 + \frac{\epsilon_{i3}}{p_{i3}} \end{pmatrix},$$

where $\epsilon_{i1}, \epsilon_{i2}$ and ϵ_{i3} represent the unpredictable part for i-th game. By default, $\epsilon_{i1} + \epsilon_{i2} + \epsilon_{i3} = 0$. Furthermore, more restrictions can be added like $|p_{ij}| > |\epsilon_{ij}|$ for $i = 1, 2, ..., n$ and $j = 1, 2, 3$. However, this is not a research interest for us at this stage.

The total change for n matches via tensor product is

$$\begin{aligned} \mathcal{E} &= \mathcal{E}(1) \otimes \mathcal{E}(2) \otimes ... \otimes \mathcal{E}(n) \\ &= \begin{pmatrix} (1 + \frac{\epsilon_{11}}{p_{11}})(1 + \frac{\epsilon_{21}}{p_{21}})...(1 + \frac{\epsilon_{n1}}{p_{n1}}) & & & & \\ & (1 + \frac{\epsilon_{11}}{p_{11}})(1 + \frac{\epsilon_{21}}{p_{21}})...(1 + \frac{\epsilon_{n2}}{p_{n2}}) & & & \\ & & (1 + \frac{\epsilon_{11}}{p_{11}})(1 + \frac{\epsilon_{21}}{p_{21}})...(1 + \frac{\epsilon_{n3}}{p_{n3}}) & & \\ & & & \ddots & \\ & & & & (1 + \frac{\epsilon_{13}}{p_{13}})(1 + \frac{\epsilon_{23}}{p_{23}})...(1 + \frac{\epsilon_{n3}}{p_{n3}}) \end{pmatrix}. \end{aligned}$$

In this circumstance, the profit of the company after the n matches is

$$
\begin{aligned}
E[C(n)] &= \sum_{\overline{x}=1}^{3^n} \Big(W_{\overline{x}} - W_{\overline{x}} \langle \overline{x} | e^{-H_e^n} | \overline{x} \rangle \langle \overline{x} | e^{-H_e^n} | \overline{x} \rangle \Big) \\
&= \sum_{\overline{x}=1}^{3^n} \Big(W_{\overline{x}} - W_{\overline{x}} \langle \overline{x} | e^{-H_p^n} | \overline{x} \rangle \langle \overline{x} | e^{-\mathcal{E} V^n} | \overline{x} \rangle \langle \overline{x} | e^{-H_e^n} | \overline{x} \rangle \Big) \\
&= \sum_{\overline{x}=1}^{3^n} \Bigg(W_{\overline{x}} - W_{\overline{x}} \Bigg[\frac{1}{2\pi} \int_0^{2\pi} (e^{i\alpha(1-\overline{x})} P_{1,1} + e^{i\alpha(2-\overline{x})} P_{2,2} + e^{i\alpha(3-\overline{x})} P_{3,3} + \ldots + e^{i\alpha(3^n-\overline{x})} P_{3^n,3^n}) d\alpha \\
&\qquad \frac{1}{2\pi} \int_0^{2\pi} (e^{i\alpha(1-\overline{x})} \mathcal{E}_{1,1} + e^{i\alpha(2-\overline{x})} \mathcal{E}_{2,2} + e^{i\alpha(3-\overline{x})} \mathcal{E}_{3,3} + \ldots + e^{i\alpha(3^n-\overline{x})} \mathcal{E}_{3^n,3^n}) d\alpha \\
&\qquad \frac{1}{2\pi} \int_0^{2\pi} (e^{i\alpha(1-\overline{x})} E_{1,1} + e^{i\alpha(2-\overline{x})} E_{2,2} + e^{i\alpha(3-\overline{x})} E_{3,3} + \ldots + e^{i\alpha(3^n-\overline{x})} E_{3^n,3^n}) d\alpha \Bigg] \Bigg).
\end{aligned}
$$

The equation above can be computed with respect to different Hamiltonian operators. This gives us a flexibility to play on. Of course, the perturbed process can be defined in more complex ways by taking seasonal weather situation, pandemic and macro economic variables, etc. into account in further research.

6. Conclusions

The betting sector is growing in a phenomenal rate, especially in the US [2], and notwithstanding the size of the industry, quantitative developments have focused on getting accurate odds or profiling customers. There is little to no work on the potential ruin of a bookmaker. Moreover there is a huge gap between the few theoretical approaches and applications in the betting sector.

In this paper, the betting process has been considered as a time series to apply the path approach with Dirac notations. We think that this kind of new approach will attract the interest of researchers in other disciplines to this sector, and more collaborative work will be produced in interdisciplinary work environments.

Furthermore, the application of such techniques can help in comparing the performance and comparative risk of different bookmakers, which could be used to better manage the financial performance of a company. Currently, the gambling market is in a phase of mergers and acquisitions at a valuation of more than one billion dollars per activity (example [32,33]). The application of more complex techniques such as those presented here could provide a relative edge when considering larger trades.

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